MATHEMATICAL PROBLEMS OF THERMOACOUSTIC TOMOGRAPHY

A Dissertation

by

LINH VIET NGUYEN

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2010

Major Subject: Mathematics

MATHEMATICAL PROBLEMS OF THERMOACOUSTIC TOMOGRAPHY

A Dissertation

by

LINH VIET NGUYEN

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Approved by:

Chair of Committee,	Peter Kuchment
Committee Members,	Brian Applegate
	Jean-Luc Guermond
	Joe Pasciak
Head of Department,	Al Boggess

August 2010

Major Subject: Mathematics

ABSTRACT

Mathematical Problems of Thermoacoustic Tomography. (August 2010)

Linh Viet Nguyen, B.S., National University of Vietnam at Ho Chi Minh City;

M.S., The University of Texas at San Antonio

Chair of Advisory Committee: Dr. Peter Kuchment

Thermoacoustic tomography (TAT) is a newly emerging modality in biomedical imaging. It combines the good contrast of electromagnetic and good resolution of ultrasound imaging. The mathematical model of TAT is the observability problem for the wave equation: one observes the data on a hyper-surface and reconstructs the initial perturbation. In this dissertation, we consider several mathematical problems of TAT. The first problem is the inversion formulas. We provide a family of closed form inversion formulas to reconstruct the initial perturbation from the observed data. The second problem is the range description. We present the range description of the spherical mean Radon transform, which is an important transform in TAT. The next problem is the stability analysis for TAT. We prove that the reconstruction of the initial perturbation from observed data is not Hölder stable if some observability condition is violated. The last problem is the speed determination. The question is whether the observed data uniquely determines the ultrasound speed and initial perturbation. We provide some initial results on this issue. They include the unique determination of the unknown constant speed, a weak local uniqueness, a characterization of the non-uniqueness, and a characterization of the kernel of the linearized operator.

To my family and T. T. H., for love and inspiration

ACKNOWLEDGMENTS

I am deeply indebted to Professor Peter Kuchment for shaping me up as a mathematician, and Dr. Mila Mogilevsky for her warm care and encouragement. I am thankful to Professors M. Agranovsky, G. Berkolaiko, A. Comech, L. Kunyansky, T. Quinto, P. Stefanov, and G. Uhlmann for the collaboration, discussion, support, and advice.

I appreciate the dedication of my committee members, including Professors B. Applegate, J.-L. Guermond, and J. Pasciak. Thanks also go to the Department of Mathematics and the Institute for Applied Mathematics and Computational Science (IAMCS) for the excellent teaching, research environment, and financial support. A special thank you is extended to Ms. Monique Stewart for her tireless work in dealing with thousands of matters.

I am grateful to my former advisor Prof. Dung Le and his wife Ms. Thu Nguyen for the kindness and support during my study at San Antonio, my uncles Nguyen Hai Trieu and Nguyen The Huong for their help during my high school years. Many friends have made my life more comfortable and enjoyable in College Station; among them are Hao Nguyen, Hieu Pham, Hoa Bui, Luke Oeding, and Yulia Hristova.

The dissertation is partly based upon the work carried out under the supports of NSF grants DMS 0604778, 0648786, and 0715090, and the grant KUS-C1-016-04 from King Abdullah University of Science and Technology (KAUST). I wish to express my gratitude to the NSF and KAUST for this support.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Mathematical foundation of tomography appeared long before the invention of the X-ray CT scanner in the 1960s by Godfrey Hounsfield and Allan McLeod Cormack, who received the 1979 Nobel Prize in Medicine. In 1917, motivated purely by mathematical interest, Radon investigated an integral transform which was later named after him. The transform sends a function on the plane to its integrals on the set of all lines. Radon was able to invert such a transform (see [51]). This has become the mathematical foundation of biomedical imaging. It also triggered the development of the new area integral geometry, later championed by I. Gelfand's school, F. John, S. Helgason, and others (e.g., [28, 24, 34]).

After the X-ray CT scanner, many other imaging modalities have been invented: Single Photon Emission Computed Tomography (SPECT), Photon Emission Tomography (PET), Magnetic Resonance Imaging (MRI), Electrical Impedance Tomography (EIT), Optical, Ultrasound, and Microwave Tomographies, among others. Along with these inventions, various related mathematical issues have been considered, such as uniqueness, inversion procedures, characterization of the perfect data, stability analysis, and regularization. These are the main issues addressed in the area inverse problems, which involves different branches of mathematics: analysis, geometry, and even combinatorics. This also determines the list of topics and mathematical tools presented in this dissertation.

Two main features of an imaging modality are its resolution and contrast. Resolution shows how well the method captures small details. Contrast is the amplitude of

---

This dissertation follows the style of *Inverse Problems*.

variation of the tissue's response to the radiation. Many imaging methods suffer from either bad resolution or bad contrast. For example, ultrasound tomography provides good resolution, but poor contrast in imaging soft biological tissues. On the other hand, the situation with the microwave tomography is just the opposite. The idea of combining different signals in a single imaging method to overcome such shortfalls has been recently proposed. Thermoacoustic tomography (TAT) is one realization of such an idea [35, 46, 47, 61, 66].

In TAT, a biological object is irradiated by a brief electromagnetic (EM) pulse in visible light or radiofrequency range. A fraction of the EM energy is absorbed by the tissues. Since EM absorption is much higher in tumors, knowing the distribution $a(x)$ of the absorbed energy would provide valuable diagnostic information. The energy absorption causes thermoelastic expansion in the tissues and thus a pressure (ultrasound) wave $u(x,t)$ propagating through the body. The ultrasound is then measured on an **observation surface** S surrounding the object (see Fig. 1). The initial pressure $f(x) = u(x,0)$ is roughly proportional to $a(x)$. One now concentrates on the recovery of $f(x)$ from the measured data $g := u|_{S\times[0,\infty)}$.

The standard mathematical model of TAT is (e.g., [12, 58, 64]):

$$\begin{cases} u_{tt}(x,t) - c^2(x) \bigtriangleup u(x,t) = 0, \ x \in \mathbb{R}^n, \ t \geq 0, \\ u(x,0) = f(x), \quad u_t(x,0) = 0, \\ u(y,t) = g(y,t), \ \text{for } y \in S, \ t \geq 0. \end{cases} \tag{1.1}$$

Here, $c(x)$ is the ultrasound speed, $g(x,t)$ is the measured data, and f is the function to be reconstructed.

While in TAT electromagnetic waves of radio frequency range are used to trigger the ultrasound signal, in the so called photo- (or opto-) acoustic tomography (PAT) [35, 46, 47, 61, 62], the frequency lies in the visual or near infra-red ranges. For

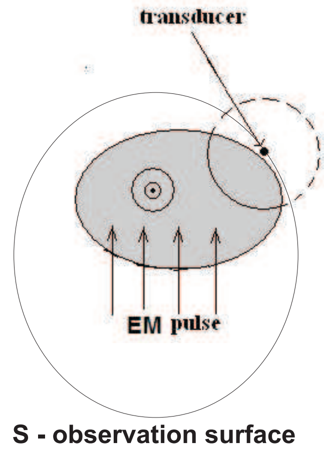

Fig. 1. Setup of TAT.

the mathematical purpose of this dissertation, there is no distinction between these methods, so we will refer to TAT only, while the results apply to PAT as well.

Mathematical problems of TAT have been intensively investigated (see reviews in [3, 23, 22, 36]). In the rest of this chapter, we briefly described the topics presented in the dissertation.

A. TAT in acoustically homogeneous medium and spherical Radon transform

Let us denote by $\mathcal{R}$ the spherical Radon [1] transform, which sends a function $f(x)$ to the function

$$\mathcal{R}(f)(x,t) := \int\limits_{S^{n-1}} f(x+t\theta)t^{n-1}dA(\theta),$$

[1]Radon's name appears here due to the similarity of this transform and the usual Radon transform.

where $dA(\theta)$ is the area measure on the unit sphere $S^{n-1} \subset \mathbb{R}^n$. If the ultrasound speed c is constant, the standard Kirchhoff-Poisson solution formulas for the wave equation (1.1) (see [11] or [15, p.77]) imply the representation:

$$u(x,t) = c_n \left[\left(\frac{\partial}{\partial t} \frac{1}{t} \right)^{\frac{n-1}{2}} \mathcal{R}f \right] (x, ct), \quad x \in \mathbb{R}^n, \quad t > 0 \tag{1.2}$$

with a constant $c_n > 0$, whose exact value will be specified when needed. The measured data g is then related to the transform $\mathcal{R}_S(f)$, which is the restriction of $\mathcal{R}(f)$ to the set of spheres centered at S.

1. Inversion formulas

Due to the formula (1.2), the reconstruction of f from g is equivalent to inverting the geometric integral operator $\mathcal{R}_S$. One might hope that the general theory of integral geometry developed by Gelfand's school [24] would provide some analytic formula for the inversion. However, their so-called kappa operator technique does not seem to work in TAT.

In TAT one often assumes that S is the unit sphere and f is supported inside S. Under these assumptions, Finch et al. [20] obtained analytic formulas in odd dimensions. Different formulas were then obtained by Finch et al [18], Xu and Wang [65], and Kunyansky [38]. In Chapter II we present a family of inversion formulas, which provides not only new formulas but also previously known ones.

2. Range description

Let $\mathcal{M}$ be the spherical mean operator which transforms function f to its mean values on the spheres:

$$\mathcal{M}(f)(x,t) := \frac{1}{|S(x,t)|} \int\limits_{S(x,t)} f(y)dA(y) = \frac{1}{\omega_n} \int\limits_{S^{n-1}} f(x+t\theta)dA(\theta).$$

Here, $S(x,t)$ is the sphere centered at x with radius t, $|S(x,t)|$ is its measure, and ω_n is the measure of the unit sphere S^{n-1}. Let $\mathcal{M}_S$ be the restriction of $\mathcal{M}$ to the set of all spheres centered on S. Then $\mathcal{M}_S$ and $\mathcal{R}_S$ are related by

$$\mathcal{M}_S(f)(x,t) = \frac{1}{\omega_n t^{n-1}} \mathcal{R}(f)(x,t).$$

Due to (1.2), the description of $h = \mathcal{M}_S(f)$ provides characterization for the measured data g, and it has been studied by many authors. The common assumptions are: S is the unit sphere and $f \in C_0^\infty(\overline{B})$, where B the unit ball enclosed by S. The complete range description, obtained by M. Agranovsky, D. Finch, and Kuchment [2], includes:

1. **Smoothness and support conditions:** $h \in C_0^\infty(S \times [0,2])$.

2. **Orthogonality conditions:**

$$\int\limits_0^2 \int\limits_S h(x,t)\partial_\nu \varphi_\lambda(x) t^{n-1} d\sigma(x)dt = 0,$$

   for any pair of eigenvalue-eigenfunction $(-\lambda^2, \varphi_\lambda)$ of the Dirichlet Laplacian Δ_D on B, where ∂_ν is the normal derivative on S.

In Chapter III we present a new proof of that description, by proving the extendibility property of some solution for Darboux equation, which is also an interesting observation by itself.

B. TAT with variable speed

There are no analytic formulas to represent the solution to the wave equation (1.1) when the speed c is not constant. However, this does not prevent progress in mathematics of TAT. Several numerical reconstruction methods have been investigated (see, e.g., [31, 55, 30]). Also, some uniqueness and stability results have been obtained in [55]. We will present in this dissertation some more results on this issue.

1. Stability analysis

Stability analysis for TAT has been considered in [20] for constant speed, and in [55] for variable speed. The main tool of the stability analysis is the propagation of singularities. The visibility condition requires all singularities of f propagate to the observation surface S. If such condition is satisfied, it was proved that the reconstruction is Lipschitz stable. In Chapter IV, we prove that the reconstruction is not even Hölder stable if the condition is violated.

2. Speed determination

Most of the recent work in TAT assumes that the ultrasound speed is known. However, in applications, it is unknown. It is natural to ask the following question:

Problem B.1 *Does the TAT data g (see (1.1)) uniquely determine $c(x)$ and $f(x)$? If not, to what extent it does?*

We present in Chapter V our initial results in Problem B.1. Work on this topic is still an ongoing project. We see some similarities between the Problem B.1 and well known unresolved transmission eigenvalue and Pompeiu problems. Exploiting these connections will be the a research direction after graduation.

C. Publications and presentations of the results

Parts of this dissertation are based on the following publications and preprints:

1. Mark Agranovsky and V. Linh Nguyen. Range conditions for a spherical mean transform and global extendibility of solutions of darboux equation. To appear in Journal d'Analyse Mathématique.

2. Yulia Hristova, Peter Kuchment, and Linh Nguyen. Reconstruction and time reversal in thermoacoustic tomography in acoustically homogeneous and inhomogeneous media. *Inverse Problems*, 24(5):055006, 25, 2008.

3. V. Linh Nguyen. On singularities and instabilty of reconstruction in thermoacoustic tomography. Preprint, http://arxiv.org/abs/0911.5521.

4. V. Linh Nguyen. A family of inversion formulas in thermoacoustic tomography. *Inverse Problems and Imaging*, 3(4):649–675, 2009.

Some results in this dissertation have been presented in the following talks:

1. Colloquial talk "Some mathematical problems of thermoacoustic tomography", Feb 25, 2010, at Department of Mathematics, University of Idaho, Moscow, Idaho.

2. Invited talk "Some mathematical problems of thermoacoustic tomography" at Inverse Problems Seminar, Feb 22, 2010, University of Washington, Seattle, Washington.

3. Invited talk "A family of inversion formulas in thermoacoustic tomography" at Conference on Applied Inverse Problems, July 20-25, 2009, University if Vienna, Vienna, Austria.

4. Contributed talk ”A family of inversion formulas in thermoacoustic tomography” at 3nd Annual Applied Mathematics Meeting for Students, March 27-28, 2009, UT Austin, Austin, Texas.

5. Expository talk ”Spherical mean Radon transform” at Graduate student organization seminar, March 26, 2009, Texas A&M University, College Station, Texas.

6. Invited talk (joint with P.Kuchment and Y. Hristova) ”Mathematical problems of thermoacoustic tomography” at the Special Session on Radon Transforms, Tomography, and Related Geometric Analysis, Amer. Math. Soc. meeting, Louisiana State University, Baton Rouge, LA, 28 - 30 March 2008.

CHAPTER II

INVERSION FORMULAS [1]

In this chapter, we present a family of closed form inversion formulas in thermoacoustic tomography in the case of a constant sound speed. The formulas are presented in both time-domain and frequency-domain versions. As special cases, they imply most of the previously known filtered backprojection type formulas.

A. Introduction to the problem and main results

It is known that soft biological tissues have low contrast with respect to ultrasound. E.g., in breast the sound speed varies not more than 10%. One thus often assumes in TAT that the ultrasound speed is constant. By choosing the proper units, the equation (1.1) of TAT becomes:

$$\begin{cases} u_{tt}(x,t) - \triangle u(x,t) = 0, \ x \in \mathbb{R}^n, \ t \geq 0, \\ u(x,0) = f(x), \quad u_t(x,0) = 0, \\ u(y,t) = g(y,t), \ \text{for } y \in S, \ t \geq 0. \end{cases} \tag{2.1}$$

To recover the image $f(x)$ from data g, one needs to invert the operator $\mathcal{T} : f \mapsto g$. Since $\mathcal{T}$ is known to be invertible from the left only, different inversion formulas exist. In this chapter, we develop a family of explicit closed form inversion formulas in the case when S is the unit sphere centered at the origin and $f \in C_0^\infty(\overline{B})$. Here B is the open unit ball enclosed by S, and $C_0^\infty(\overline{B})$ is the set of all functions $f \in C^\infty(\mathbb{R}^n)$ supported inside $\overline{B}$.

We will also need to deal with some other operators closely related to $\mathcal{T}$. Consider

[1] Reprinted with permission from "A family of inversion formulas in thermoacoustic tomography", by V. Linh Nguyen, Inverse Problems and Imaging, 3(4):649–675, 2009.

the wave equation problem, which is only different from (2.1) in the initial conditions:

$$\begin{cases} u_{tt}(x,t) - \triangle u(x,t) = 0, \quad x \in \mathbb{R}^n, \quad t \geq 0, \\ u(x,0) = 0, \quad u_t(x,0) = f(x), \\ u(y,t) = g(y,t), \text{ for } y \in S, t \geq 0. \end{cases} \tag{2.2}$$

One can now define the operator $\mathcal{P}$ that maps function f into g: $\mathcal{P}(f) = g$. Then $\mathcal{P}$ is related to $\mathcal{T}$ as follows:

$$\mathcal{P}(f)(y,t) = \int_0^t \mathcal{T}(f)(y,\tau)d\tau \text{ and } \mathcal{T}(f)(t) = \partial_t \mathcal{P}(f)(t).$$

We also introduce the spherical Radon transform $\mathcal{R}_S$ with centers on S by

$$\mathcal{R}_S(f)(y,t) = \int_{S^{n-1}} f(y + t\omega)t^{n-1}d\sigma(\omega), \text{ for all } (y,t) \in S \times \mathbb{R}_+.$$

Finally, $\mathcal{M}_S$ is the spherical mean operator with centers on S:

$$\mathcal{M}_S(f)(y,t) = \frac{1}{\omega_n} \int_{S^{n-1}} f(y + t\omega)d\sigma(\omega), \text{ for all } (y,t) \in S \times \mathbb{R}_+.$$

In these formulas, $d\sigma(\omega)$ is the standard surface measure on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ and ω_n is the total measure of the sphere.

In various works cited below, different operators from this list were considered. However, due to the known explicit connections between $\mathcal{T}$ and $\mathcal{P}$, $\mathcal{R}_S$, and $\mathcal{M}_S$ (e.g., [11, 34]), inversion formulas for these operators are closely related.

The first such formulas, in odd dimensions, were obtained by Finch, Patch and Rakesh [20] using some trace identities for the wave operator. Xu and Wang [65] derived a different formula for $n = 3$ by working in the frequency domain. A formula for all dimensions, which coincides with that of [65] when $n = 3$, was presented by Kunyansky [38]. Its derivation is based upon some symmetry relation for special functions. By applying the same method as in [20], Finch, Haltmeier and Rakesh [18] obtained inversion formulas for even dimensions, which involve the data measured for an infinite time period. The authors of [18] also derived another type of inversion formulas for even n, which uses only the data measured for a finite period of time (which we will refer as "finite-time formulas for even dimensions").

In spite of availability of several types of closed form inversion formulas, some questions have remained unanswered. For instance, it was not clear, what is the relation, if any, between formulas of [18, 20] and [38, 65], (which are known to be not equivalent outside the range of the operator $\mathcal{T}$ which maps $C_0^\infty(\overline{B})$ to $C^\infty(S\times[0,\infty))$. The same applies to the two types of inversion formulas derived in [18] for even dimensions.

The goal of this chapter is to obtain a unified family of closed form inversion formulas, which would produce formulas of [18, 20, 38, 65] as particular cases.

We now formulate the main results. Consider the function

$$G(s,\lambda) = \frac{i}{4}\left(\frac{\lambda}{2\pi s}\right)^{\frac{n-2}{2}} H^{(1)}_{\frac{n-2}{2}}(\lambda s), \tag{2.3}$$

where $H^{(1)}_{\frac{n-2}{2}}$ is the Hankel function of the first kind. The following facts are well known (e.g.,[1]):

- $\Phi(x,y,\lambda) = G(|x-y|,\lambda)$ is the solution for the Helmholtz equation

  $$\Delta U(y) + \lambda^2 U(y) = -\delta(y-x), \tag{2.4}$$

  obtained by limiting absorption.

- For odd n and any $s>0$, $G(s,.)\in C^\infty(\mathbb{R})$. For even n, $G(s,.)\in C^\infty(\mathbb{R}\setminus\{0\})$ with a logarithmic singularity at zero

- For fixed $s > 0$, G has the following asymptotic behavior:

$$G(s,\lambda) = \mathcal{O}(|\lambda|^{\frac{n-3}{2}}), \text{ as } |\lambda| \longrightarrow \infty.$$

Let $g(y,t)$ be the function in (2.1) that represents the TAT data. Since $f \in C_0^\infty(\overline{B})$, using the Kirchhoff-Poisson solution formulas [15], we see that $g \in C^\infty(S \times \mathrm{I\!R})$ and g vanishes to infinite order at $t = 0$. Moreover, if n is odd then g is compactly supported. If n is even, it can be shown that (e.g., [60]):

$$\left\|\partial_t^k g(.,t)\right\|_{L^\infty(S)} \leq C\eta(t)\|f\|_{L^2(\mathrm{I\!R}^n)}, \tag{2.5}$$

where η decays as fast as t^{-n-k} as $t \to \infty$.

Let $g_0(y,t)$ be the even extension of g with respect to t and $\hat{g}_0$ be its time Fourier transform:

$$\hat{g}_0(y,\lambda) = \frac{1}{2\pi}\int\limits_{\mathrm{I\!R}} g_0(y,t)e^{i\lambda t}dt.$$

Due to the aforementioned properties of g, $\hat{g}_0$ is continuous on $S \times \mathrm{I\!R}$ and decays faster than any powers of λ as $|\lambda| \to \infty$.

Our family of inversion formula reads as follows [2]:

Theorem A.1 *Suppose that $f \in C_0^\infty(\overline{B})$ and $g = \mathcal{T}(f)$. Then for any $x \in B$ and $\xi \in \mathrm{I\!R}^n$, the following equality holds*

$$f(x) = -2\int\limits_S \left(\frac{d}{ds}\int\limits_{\mathrm{I\!R}} \overline{G}(s,\lambda)\hat{g}_0(y,\lambda)d\lambda\right)\Bigg|_{s=|x-y|} \frac{\langle y-x, y-\xi\rangle}{|x-y|}d\sigma(y), \tag{2.6}$$

where $\overline{G}$ is the complex conjugate of G.

One can rewrite this inversion formula without going to the frequency domain.

[2]The inner improper integral of (2.6) is convergent. Indeed, due to the aforementioned properties of G and $\hat{g}_0$, the function $\overline{G}(s,\lambda)\hat{g}_0(y,\lambda)$ decays faster than any powers of λ as $\lambda \to \pm\infty$, and has at most the logarithmic singularity at $\lambda = 0$.

Namely, we define a transform $\mathcal{W}$ of a function $v \in C^\infty[0,\infty)$ as follows (as long as the expressions involved make sense)[3]:

$$\mathcal{W}(v)(s) := \begin{cases} c_n \left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}} \int\limits_s^\infty \frac{v(t)}{\sqrt{t^2-s^2}} dt, \text{ if } n \text{ is even,} \\ c_n \left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-3}{2}} \left(\frac{v(s)}{s}\right), \text{ if } n \text{ is odd.} \end{cases} \tag{2.7}$$

Here

$$c_n := \begin{cases} \frac{(-1)^{\frac{n-2}{2}}}{(2\pi)^{\frac{n}{2}}}, \text{ if } n \text{ is even,} \\ \frac{(-1)^{\frac{n-3}{2}}}{2(2\pi)^{\frac{n-1}{2}}}, \text{ if } n \text{ is odd.} \end{cases} \tag{2.8}$$

One can now obtain another representation of the inner integral in (2.6) (see Lemma C.1):

$$\int\limits_{\mathbb{R}} \overline{G}(s,\lambda)\hat{g}_0(y,\lambda)d\lambda = \mathcal{W}(g)(y,s), \tag{2.9}$$

where $\mathcal{W}(g)(y,s) := \mathcal{W}(g_y)(s)$ with $g_y(t) := g(y,t)$. Then Theorem A.1 is equivalent to

Theorem A.2 *Suppose that $f \in C_0^\infty(\overline{B})$ and $g = \mathcal{T}(f)$. Then for any $x \in B$ and $\xi \in \mathbb{R}^n$, the following equality holds*

$$f(x) \quad = \quad -2\int\limits_S \left(\frac{d}{ds}\mathcal{W}(g)(y,s)\right)\Bigg|_{s=|x-y|} \frac{\langle y-x, y-\xi\rangle}{|x-y|} d\sigma(y). \tag{2.10}$$

One can look at the inner integral in (2.6) in a different manner yet. Namely, due to the relation between $\mathcal{T}$ and $\mathcal{R}_S$, it can be shown (see Lemma C.3) that

$$\int\limits_{\mathbb{R}} \overline{G}(s,\lambda)\hat{g}_0(y,\lambda)d\lambda = \frac{2}{\pi}\int\limits_0^\infty \lambda\mathcal{R}(s,\lambda)\int\limits_0^\infty \mathcal{R}_S(f)(y,r)\mathcal{I}(r,\lambda)drd\lambda, \tag{2.11}$$

where $\mathcal{R}$ and $\mathcal{I}$ are the real and imaginary parts of G. Then Theorem A.1 implies

---

[3]As noted in Section G, $\mathcal{W}$ is a known transform that intertwines the second derivative and the Bessel operator.

Theorem A.3 *Suppose that* $f \in C_0^\infty(\overline{B})$ *and* $g = \mathcal{R}_S(f)$. *Then for any* $x \in B$ *and* $\xi \in \mathbb{R}^n$, *the following equality holds*

$$f(x) = -\frac{4}{\pi} \int\limits_S \left(\frac{d}{ds} K_n(y,s) \right)_{s=|x-y|} \frac{\langle y-x, y-\xi \rangle}{|x-y|} d\sigma(y), \tag{2.12}$$

where

$$K_n(y,s) = \int\limits_0^\infty \lambda \mathcal{R}(s,\lambda) \int\limits_0^\infty g(y,r) \mathcal{I}(r,\lambda) dr d\lambda.$$

The key ingredient in the proof of Theorem A.1 is the following identity, which is equivalent to a range description of operator $\mathcal{T}$ (see Remark D.3):

Theorem A.4 *Suppose that* Ω *is a ball,* $f \in C_0^\infty(\overline{\Omega})$, *and* u *solves (2.1). Let* u_0 *be the even extension of* u *with respect to* t. *Then for all* $x \in \Omega$ *we have*

$$\int\limits_{\partial\Omega} \int\limits_{\mathbb{R}} \overline{G}(|x-y|,\lambda) \hat{u}_0(y,\lambda) d\lambda d\sigma(y) = 0. \tag{2.13}$$

This statement implies that adding to any inversion formula an expression

$$A \int\limits_S \int\limits_{\mathbb{R}} \overline{G}(|x-y|,\lambda) \hat{g}_0(y,\lambda) d\lambda d\sigma(y)$$

with an arbitrary linear operator A also produces an inversion formula (since the added term vanishes on the range of the operator to be inverted). In particular, if A is the operator of multiplication by an arbitrary function $\varphi(x)$, one can add the expression

$$\varphi(x) \int\limits_S \int\limits_{\mathbb{R}} \overline{G}(|x-y|,\lambda) \hat{g}_0(y,\lambda) d\lambda d\sigma(y).$$

to (2.10) and (2.12) to obtain the following inversion formulas (we use the identities (2.9) and (2.11) again):

Corrolary A.5 *Suppose that* $f \in C_0^\infty(\overline{B})$ *and* $g = \mathcal{T}(f)$. *Let* φ *be an arbitrary* *function defined on B. Then for any $x \in B$ and $\xi \in \mathbb{R}^n$, the following equalities hold*

$$\begin{aligned} f(x) &= -\frac{4}{\pi}\int\limits_S \left(\frac{d}{ds}K_n(y,s)\right)_{s=|x-y|} \frac{\langle y-x, y-\xi\rangle}{|x-y|} d\sigma(y) \\ &+ \varphi(x)\int\limits_S K_n(y,|x-y|)d\sigma(y), \end{aligned} \tag{2.14}$$

$$\begin{aligned} f(x) &= -2\int\limits_S \left(\frac{d}{ds}\mathcal{W}(g)(y,s)\right)\Big|_{s=|x-y|} \frac{\langle y-x, y-\xi\rangle}{|x-y|} d\sigma(y) \\ &+ \varphi(x)\int\limits_S \mathcal{W}(g)(y,|x-y|)d\sigma(y). \end{aligned} \tag{2.15}$$

Now, by suitable choices of ξ and φ in (2.14) and (2.15), one can recover the inversion formulas known in the literature. If we let $\xi = x$, and $\varphi = -2(n-2), -2(n-1)$, or $-2n$ in (2.15), we obtain correspondingly the formulas derived in [18, Theorem 1.5] and [20, Theorem 3]:

Proposition A.6 *Let $f \in C_0^\infty(\overline{B})$. Then for any $x \in B$:*

$$f(x) = -2\left(\mathcal{P}^* t\partial_t^2 \mathcal{P} f\right)(x) \tag{2.16}$$

$$f(x) = -2\left(\mathcal{P}^* \partial_t t\partial_t \mathcal{P} f\right)(x) \tag{2.17}$$

$$f(x) = -2\left(\mathcal{P}^* \partial_t^2 t \mathcal{P} f\right)(x) \tag{2.18}$$

Here $\partial_t = \frac{d}{dt}$ is the derivative with respect to t, and $\mathcal{P}^*$ is the L^2-adjoint of $\mathcal{P}$.

Choosing $\xi = x$ and $\varphi = 0$ in (2.14), one gets a finite-time inversion formula for even dimensions similar to the second one in [18, Theorem 1.3]:

Proposition A.7 *Assume that n is even, $f \in C_0^\infty(\overline{B})$, and $g = \mathcal{M}_S(f)$. Then for all $x \in B$, $f(x)$ is equal to*

$$\frac{(-1)^{\frac{n-2}{2}}\omega_n}{(2\pi)^n}\int\limits_S\int\limits_0^2 \left[\partial_r r\left(\partial_r \frac{1}{r}\right)^{n-1} r^{n-1} g\right](y,r)\ln\left|r^2-|x-y|^2\right| dr d\sigma(y),$$

where ω_n is the surface area of the unit sphere S^{n-1}.

In particular, when $n = 2$, we obtain the second formula in [18, Theorem 1.1]:

Corrolary A.8 *Assume that $n = 2$, $f \in C_0^\infty(\overline{B})$, and $g = \mathcal{M}_S(f)$. Then for any $x \in B$,*

$$f(x) = \frac{1}{2\pi} \int\limits_S \int\limits_0^2 [\partial_r r \partial_r g]\,(y,r) \ln \left| r^2 - |x-y|^2 \right| dr d\sigma(y). \tag{2.19}$$

Finally, let

$$J(s) = \frac{J_{\frac{n-2}{2}}(s)}{s^{\frac{n-2}{2}}}, \quad N(s) = \frac{N_{\frac{n-2}{2}}(s)}{s^{\frac{n-2}{2}}},$$

where $J_{\frac{n-2}{2}}$ and $N_{\frac{n-2}{2}}$ are the Bessel and Neumann functions of order $\frac{n-2}{2}$. Choosing $\xi = 0$ and $\varphi = 0$ in (2.14), one arrives at:

Proposition A.9 *Assume that $f \in C_0^\infty(\overline{B})$ and $g = \mathcal{R}_S(f)$. Then for any $x \in B$,*

$$f(x) = \frac{-1}{2(2\pi)^{n-1}} \nabla_x \cdot \int\limits_S n(y) k_n(y, |x-y|) d\sigma(y). \tag{2.20}$$

where $n(y)$ is the outward normal of S at y, and

$$k_n(y,s) \quad = \quad \int\limits_0^\infty \lambda^{2n-3} N(s\lambda) \int\limits_0^2 g(y,r) J(r\lambda) dr d\lambda.$$

This inversion formula is equivalent to the one obtained in [38]. Indeed, let

$$\begin{aligned} h(y,s) \quad &= \quad \int\limits_0^\infty \lambda^{2n-3} N(s\lambda) \int\limits_0^2 g(y,r) J(r\lambda) dr d\lambda \\ &- \quad \int\limits_0^\infty \lambda^{2n-3} J(s\lambda) \int\limits_0^2 g(y,r) N(r\lambda) dr d\lambda. \end{aligned}$$

It can be shown that $h(y,s) = 2k_n(y,s)$, and thus Proposition A.9 implies the following result of [38]:

Proposition A.10 *Assume that $f \in C_0^\infty(\overline{B})$ and $g = \mathcal{R}_S(f)$. Then for any $x \in B$,*

$$f(x) = \frac{-1}{4(2\pi)^{n-1}} \nabla_x \cdot \int\limits_S n(y) h(y, |x-y|) d\sigma(y). \tag{2.21}$$

Remark A.11 *Formulas of [18, 20, 38, 65] do not reconstruct $f(x)$ inside S correctly if a part of support of f lies outside S. This feature is absent in other reconstruction methods such as time reversal (the readers are referred to [3, 36] for these discussion). It would be interesting to see whether formulas contained in Corollary A.5 are any different in this regard.*

The chapter is organized as follows. In Section B, we derive Theorem A.1 from Theorem A.4. In Section C, we show that Theorem A.2 equivalent to Theorem A.1 and Theorem A.3 is implied by Theorem A.1. Two proofs of Theorem A.4 are presented in Section D. In Section E, Propositions A.6, A.7, A.9 and A.10 are derived from Corollary A.5 using the aforementioned choices of ξ and φ. Proofs of some auxiliary results are given in Section F. Finally, some remarks are provided in Section G.

B. Derivation of Theorem A.1 from Theorem A.4

Let u solve (2.1) and u_0 be its even extension with respect to t. Then u_0 solves the wave equation on $\mathbb{R}^n \times \mathbb{R}$. Therefore,

$$\lambda^2 \hat{u}_0(y,\lambda) + \Delta \hat{u}_0(y,\lambda) = 0. \tag{2.22}$$

Due to (2.4) and (2.22), we have the Green's identity:

$$\hat{u}_0(x,\lambda) = -\int\limits_S \left[\frac{\partial \overline{G}(|x-y|,\lambda)}{\partial \nu_y} \hat{u}_0(y,\lambda) - \frac{\partial \hat{u}_0(y,\lambda)}{\partial \nu_y} \overline{G}(|x-y|,\lambda) \right] d\sigma(y). \tag{2.23}$$

Since $f(x) = u_0(x,0)$, the Fourier inversion gives

$$f(x) = \int\limits_{\mathbb{R}} \hat{u}_0(x,\lambda) d\lambda.$$

Due to (2.23), we get

$$\begin{aligned} f(x) &= -\int_S \int_{\mathbb{R}} \frac{\partial \overline{G}(|x-y|,\lambda)}{\partial \nu_y} \hat{u}_0(y,\lambda) d\lambda d\sigma(y) \\ &+ \int_S \int_{\mathbb{R}} \frac{\partial \hat{u}_0(y,\lambda)}{\partial \nu_y} \overline{G}(|x-y|,\lambda) d\lambda d\sigma(y). \end{aligned} \tag{2.24}$$

Let $S(0,r)$ be the sphere centered at the origin with radius r. For any function $H \in C^1(\mathbb{R}^n)$, by changing variables, we derive

$$\frac{d}{dr}\left(r^{-n+1} \int_{S(0,r)} H(y) d\sigma(y) \right)\Bigg|_{r=1} = \frac{d}{dr}\left(\int_S H(ry) d\sigma(y) \right)\Bigg|_{r=1}.$$

That is,

$$\frac{d}{dr}\left(r^{-n+1} \int_{S(0,r)} H(y) d\sigma(y) \right)\Bigg|_{r=1} = \int_S \frac{\partial H}{\partial \nu_y}(y) d\sigma(y).$$

Applying this equality for

$$H(y) = \int_{\mathbb{R}} \overline{G}(|y-x|,\lambda) \hat{u}_0(y,\lambda) d\lambda,$$

we get

$$\begin{aligned} &\int_S \frac{\partial}{\partial \nu_y} \int_{\mathbb{R}} \overline{G}(|y-x|,\lambda) \hat{u}_0(y,\lambda) d\lambda d\sigma(y) = \\ &\frac{d}{dr}\left(r^{-n+1} \int_{S(0,r)} \int_{\mathbb{R}} \overline{G}(|y-x|,\lambda) \hat{u}_0(y,\lambda) d\lambda d\sigma(y) \right)\Bigg|_{r=1}. \end{aligned}$$

Due to Theorem A.4, the right hand side is zero. Thus,

$$\int_S \frac{\partial}{\partial \nu_y} \int_{\mathbb{R}} \overline{G}(|y-x|,\lambda) \hat{u}_0(y,\lambda) d\lambda d\sigma(y) = 0.$$

Therefore,

$$\int_S \int_{\mathbb{R}} \frac{\partial \overline{G}(|y-x|,\lambda)}{\partial \nu_y} \hat{u}_0(y,\lambda) d\lambda d\sigma(y)$$
$$= -\int_S \int_{\mathbb{R}} \overline{G}(|y-x|,\lambda) \frac{\partial \hat{u}_0}{\partial \nu_y}(y,\lambda) d\lambda d\sigma(y). \tag{2.25}$$

Combining this and (2.24), we get

$$\begin{aligned} f(x) &= -2\int_S \int_{\mathbb{R}} \frac{\partial \overline{G}(|x-y|,\lambda)}{\partial \nu_y} \hat{u}_0(y,\lambda) \\ &= -2\int_S \int_{\mathbb{R}} \overline{G}_s(|x-y|,\lambda) \hat{u}_0(y,\lambda) \frac{\langle y-x, y\rangle}{|x-y|}. \end{aligned} \tag{2.26}$$

Here $\overline{G}_s(s,\lambda)$ is the derivative of $\overline{G}(s,\lambda)$ with respect to s.

Applying Theorem A.4 for $\Omega = B$, we get

$$\int_S \int_{\mathbb{R}} \overline{G}(|x-y|,\lambda) \hat{u}_0(y,\lambda) = 0. \tag{2.27}$$

Taking the derivative with respect to x of the above identity along the direction ξ, we obtain

$$-2\int_S \int_{\mathbb{R}} \overline{G}_s(|x-y|,\lambda) \hat{u}_0(y,\lambda) \frac{\langle y-x, \xi\rangle}{|x-y|} = 0.$$

Subtracting this equality from (2.26), we conclude that

$$\begin{aligned} f(x) &= -2\int_S \int_{\mathbb{R}} \overline{G}_s(|x-y|,\lambda) \hat{u}_0(y,\lambda) \frac{\langle y-x, y-\xi\rangle}{|x-y|} d\lambda d\sigma(y) \\ &= -2\int_S \left(\frac{d}{ds} \int_{\mathbb{R}} \overline{G}(s,\lambda) \hat{g}_0(y,\lambda) d\lambda \right) \Bigg|_{s=|x-y|} \frac{\langle y-x, y-\xi\rangle}{|x-y|} d\sigma(y). \end{aligned}$$

Theorem A.1 is proved.

C. Derivation of Theorems A.2 and A.3

We prove Theorem A.2 by showing that it is equivalent to Theorem A.1. Indeed, it suffices to prove the following result:

Lemma C.1 *Suppose that* $v \in L^1(\mathbb{R}) \cap C[0,\infty)$ *such that* $\hat{v}$ *has proper decay at infinity, say* $\hat{v}(\lambda)$ *decays faster than any powers of* λ *as* $|\lambda| \to \infty$. *Then for any* $s > 0$,

$$\int_{\mathbb{R}} \overline{G}(s,\lambda)\hat{v}(\lambda)d\lambda = \mathcal{W}(v)(s). \tag{2.28}$$

In order to prove this lemma, we need the following explicit formula for G:

Proposition C.2 *Let* c_n *be as in (2.8). For any* $s > 0$, *we have*

$$G(s,\lambda) = \begin{cases} c_n \left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}} \left(\int\limits_s^\infty \frac{e^{i\lambda t}}{\sqrt{t^2-s^2}}dt\right), & \textit{if } n \textit{ is even,} \\ c_n \left[\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-3}{2}} \frac{e^{i\lambda s}}{s}\right], & \textit{if } n \textit{ is odd.} \end{cases} \tag{2.29}$$

This formula must be well known. For completeness, its proof is given in Section F.

Proof of Lemma C.1 The relation (2.29) means that $G(s,\lambda) = \mathcal{W}(e^{i\lambda t})(s)$. Keeping this fact in mind, one sees that the following proof consists of just changing the order of the operator $\mathcal{W}$ with the integral sign:

- For even n, due to the decay of $\hat{v}$ and Proposition C.2, the following calculations are valid:

$$\begin{aligned} \int_{\mathbb{R}} \overline{G}(s,\lambda)\hat{v}(\lambda)d\lambda &= c_n \int_{\mathbb{R}} \left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}} \left(\int_s^\infty \frac{e^{-i\lambda t}}{\sqrt{t^2-s^2}}dt\right)\hat{v}(\lambda)d\lambda \\ &= c_n \left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}} \int_s^\infty \frac{1}{\sqrt{t^2-s^2}} \left(\int_{\mathbb{R}} e^{-i\lambda t}\hat{v}(\lambda)d\lambda\right)dt \\ &= c_n \left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}} \int_s^\infty \frac{v(t)}{\sqrt{t^2-s^2}}dt = \mathcal{W}(v)(s). \end{aligned}$$

- For an odd n, due to Proposition C.2 and decay of $\hat{v}$,

$$\begin{aligned}\int\limits_{\mathbb{R}} \overline{G}(s,\lambda)\hat{v}(\lambda)d\lambda &= c_n \int\limits_{\mathbb{R}} \left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-3}{2}} \left(\frac{e^{-is\lambda}}{s}\right)\hat{v}(\lambda)d\lambda \\ &= c_n \left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-3}{2}} \left(\int\limits_{\mathbb{R}} \frac{e^{-is\lambda}}{s}\hat{v}(\lambda)d\lambda\right) \\ &= c_n \left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-3}{2}} \left(\frac{v(s)}{s}\right) = \mathcal{W}(v)(s).\end{aligned}$$

The lemma is proved.∎

We now prove the following result, which shows that Theorem A.1 implies Theorem A.3:

Lemma C.3 *Let $f \in C_0^\infty(\mathbb{R}^n)$, $g = \mathcal{T}(f)$ and g_0 be its even extension with respect to t. Then*

$$\int\limits_{\mathbb{R}} \overline{G}(s,\lambda)\hat{g}_0(y,\lambda)d\lambda = \frac{2}{\pi}\int\limits_0^\infty \lambda\mathcal{R}(s,\lambda)\int\limits_0^\infty \mathcal{R}_S(f)(y,r)\mathcal{I}(r,\lambda)dr.$$

We need the following auxiliary result:

Proposition C.4 *Let $f \in C_0^\infty(\mathbb{R}^n)$ and $g = \mathcal{T}(f)$. Then*

$$\int\limits_0^\infty g(y,r)e^{i\lambda r}dr = -i\lambda\int\limits_0^\infty \mathcal{R}_S(f)(y,r)G(r,\lambda)dr.$$

Proof For the expository purpose, we provide here two proofs of this proposition.

1) Let $\bar{u}(x,\lambda) = \int\limits_0^\infty u(x,s)e^{i\lambda s}ds$. Due to (2.1), $\bar{u}(.,\lambda)$ is the solution for

$$\Delta U(x) + \lambda^2 U(x) = i\lambda f(x),$$

obtained by limiting absorption. Due to (2.4), we obtain

$$\bar{u}(y,\lambda) = -i\lambda\int\limits_{\mathbb{R}^n} f(x)G(|y-x|,\lambda)dx$$

$$= \quad -i\lambda \int_0^\infty (\mathcal{R}_S f)(y,r) G(r,\lambda) dr.$$

This proves the proposition.

2) Consider the transform

$$\mathcal{B}(v)(r) = \begin{cases} (-1)^{\frac{n-2}{2}} c_n \frac{d}{dr} \left(\frac{1}{r}\frac{d}{dr}\right)^{\frac{n-2}{2}} \left(\int_0^r \frac{v(t)}{\sqrt{r^2-t^2}} dt\right), & \text{if } n \text{ is even,} \\ (-1)^{\frac{n-3}{2}} c_n \frac{d}{dr} \left(\frac{1}{r}\frac{d}{dr}\right)^{\frac{n-3}{2}} \left(\frac{v(r)}{r}\right), & \text{if } n \text{ is odd} \end{cases}$$

Let $g \in C^\infty(S \times [0,\infty))$, we also define $\mathcal{B}(g)(y,r) = \mathcal{B}(g_y)(r)$ where $g_y(r) = g(y,r)$. Let $g = \mathcal{T}(f)$, we then have (e.g., [11, 15]) $g = \mathcal{B}(\mathcal{R}_S f)$. Therefore,

$$\int_0^\infty g(y,r) e^{i\lambda r} dr = \int_0^\infty \mathcal{B}(\mathcal{R}_S f)(y,r) e^{i\lambda r} dr.$$

Substituting the expression of $\mathcal{B}$ into the integral and integrating by parts, we obtain

$$\int_0^\infty g(y,r) e^{i\lambda r} dr = -i\lambda \int_0^\infty (\mathcal{R}_S f)(y,r) \mathcal{W}(e_\lambda)(r) dr,$$

where $e_\lambda(r) = e^{i\lambda r}$. Due to Proposition C.2, we have

$$\int_0^\infty g(y,r) e^{i\lambda r} dr = -i\lambda \int_0^\infty (\mathcal{R}_S f)(y,r) G(r,\lambda) dr.$$

This completes the proof.

∎

Proof of Lemma C.3 Due to (2.29), $G(s,-\lambda) = \overline{G}(s,\lambda)$. Since $\hat{g}_0$ is even, we have

$$\int_{\mathbb{R}} \overline{G}(s,\lambda) \hat{g}_0(y,\lambda) d\lambda = 2 \int_0^\infty \mathcal{R}(s,\lambda) \hat{g}_0(y,\lambda) d\lambda. \tag{2.30}$$

Since g_0 is the even extension of g with respect to t,

$$\hat{g}_0(y,\lambda) = \frac{1}{2\pi}\int\limits_{\mathbb{R}} g_0(y,t)e^{i\lambda t}dt = \frac{1}{\pi} Re\int\limits_0^\infty g(y,r)e^{i\lambda r}dr.$$

Due to Proposition C.4, we get

$$\begin{aligned}\hat{g}_0(y,\lambda) &= \frac{1}{\pi} Re\left(-i\lambda\int\limits_0^\infty \mathcal{R}_S(f)(y,r)G(r,\lambda)dr\right)\\ &= \frac{\lambda}{\pi}\int\limits_0^\infty \mathcal{R}_S(f)(y,r)\mathcal{I}(r,\lambda)dr.\end{aligned}$$

From (2.30), we arrive at

$$\int\limits_{\mathbb{R}} \overline{G}(s,\lambda)\hat{g}_0(y,\lambda)d\lambda = \frac{2}{\pi}\int\limits_0^\infty \lambda\mathcal{R}(s,\lambda)\int\limits_0^\infty \mathcal{R}_S(f)(y,r)\mathcal{I}(r,\lambda)dr.$$

The proof is completed.∎

D. Proof of Theorem A.4

In this section, we present two proofs of Theorem A.4. One of them relies on some results of [18, 20] and works in the time domain while the other one is self-contained and works in the frequency domain. Without loss of generality, we can assume that $\Omega = B$, the open unit ball. We need to prove that if $f \in C^\infty(\overline{B})$ and $g = \mathcal{T}(f)$ then for all $x \in B$

$$\int\limits_S\int\limits_{\mathbb{R}} \overline{G}(|x-y|,\lambda)\hat{g}_0(y,\lambda)d\lambda d\sigma(y) = 0, \tag{2.31}$$

where g_0 is the even extension of g with respect to t.

1. The indirect proof

This proof is somewhat indirect since it uses inversion formulas in [18, 20].

Due to Lemma C.1, equality (2.31) is equivalent to

$$\int_S \mathcal{W}(g_0)(y,|x-y|)d\sigma(y) = 0,$$

for all $x \in B$. Or, since $g = g_0|_{S\times[0,\infty)}$,

$$\int_S \mathcal{W}(g)(y,|x-y|)d\sigma(y) = 0. \tag{2.32}$$

We introduce the following definition

Definition D.1 *Let $\widetilde{C}(S\times[0,\infty))$ be the space of all functions $h \in C^\infty(S\times[0,\infty))$ satisfying the following conditions:*

i) h vanishes at $t = 0$ to infinite order.

ii) If n is odd then h is compactly supported. If n is even then for any nonnegative integer k, $\|\partial_t^k h(.,t)\|_{L^\infty(S)} = \mathcal{O}(t^{-n-k+1})$ as $t \to \infty$.

Then the operator $\mathcal{P}$, introduced in Section A, maps $C_0^\infty(\overline{B})$ into $\widetilde{C}(S\times[0,\infty))$. Indeed, for $f \in C_0^\infty(\overline{B})$, the fact that $\mathcal{P}(f)$ satisfies the vanishing condition $i)$ can be shown by using the Kirchhoff-Poisson solution formulas for wave equation (e.g., [15]). The decay property $ii)$ of $\mathcal{P}(f)$ follows from [60].

We now prove the following auxiliary result:

Lemma D.2 *Let $x \in B$ and $h \in \widetilde{C}(S\times[0,\infty))$. Then*

$$\mathcal{P}^*(h)(x) = \int_S \mathcal{W}(h)(y,|x-y|)d\sigma(y),$$

where $\mathcal{P}^*$ is the L^2-adjoint of $\mathcal{P}$.

Proof For an even n, a direct calculation (see [22]) gives

$$\mathcal{P}^*(h)(x) = c_n \int_S \int_{|x-y|}^{\infty} \frac{\left(\frac{d}{dt}\frac{1}{t}\right)^{\frac{n-2}{2}} h(y,t)}{\sqrt{t^2-|x-y|^2}} dt dy. \tag{2.33}$$

We observe that

$$\begin{aligned}
\frac{1}{s}\frac{d}{ds}\int_s^\infty \frac{h(y,t)}{\sqrt{t^2-s^2}}dt &= \frac{1}{s}\frac{d}{ds}\int_s^\infty \frac{h(y,t)}{t}\frac{t}{\sqrt{t^2-s^2}}dt \\
&= -\frac{1}{s}\frac{d}{ds}\int_s^\infty \frac{d}{dt}\left(\frac{h(y,t)}{t}\right)\sqrt{t^2-s^2}dt \\
&= -\int_s^\infty \frac{d}{dt}\left(\frac{h(y,t)}{t}\right)\left(\frac{1}{s}\frac{d}{ds}\right)\left(\sqrt{t^2-s^2}\right)dt \\
&= \int_s^\infty \frac{d}{dt}\left(\frac{h(y,t)}{t}\right)\frac{1}{\sqrt{t^2-s^2}}dt.
\end{aligned}$$

Hence, by induction, for all $k \geq 0$,

$$\int_s^\infty \frac{\left(\frac{d}{dt}\frac{1}{t}\right)^k h(y,t)}{\sqrt{t^2-s^2}}dt = \left(\frac{1}{s}\frac{d}{ds}\right)^k \int_s^\infty \frac{h(y,t)}{\sqrt{t^2-s^2}}dt.$$

Therefore, due to (2.33), we have

$$\begin{aligned}
\mathcal{P}^*(h)(x) &= c_n \int_S \left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}} \int_s^\infty \frac{h(y,t)}{\sqrt{t^2-s^2}}dtdy \\
&= \int_S \mathcal{W}(h)(y,|x-y|)d\sigma(y).
\end{aligned}$$

For an odd n, a direct calculation (see [20]) gives

$$\begin{aligned}
\mathcal{P}^*(g)(x) &= c_n \int_S \left(\frac{1}{t}\frac{d}{dt}\right)^{\frac{n-3}{2}} \left(\frac{h(y,t)}{t}\right)\bigg|_{t=|y-x|} d\sigma(y) \\
&= \int_S \mathcal{W}(h)(y,|x-y|)d\sigma(y).
\end{aligned}$$

This proves the lemma. ∎

This lemma tells us that (2.32) is equivalent to $\mathcal{P}^*(g) = 0$. Equivalently,

$$\mathcal{P}^*\partial_t\mathcal{P}(f) = 0. \tag{2.34}$$

We now derive this equality from inversion formulas in [18, 20]. Indeed, if n is even, [18, Theorem 1.5] gives

$$\begin{aligned} f(x) &= -2\left(\mathcal{P}^* t\partial_t^2 \mathcal{P} f\right)(x), \\ f(x) &= -2\left(\mathcal{P}^* \partial_t t \partial_t \mathcal{P} f\right)(x). \end{aligned}$$

By subtracting these two inversion formulas we obtain the desired equality (2.34). If n is odd, [20, Theorem 3] gives

$$\begin{aligned} f(x) &= -2\left(\mathcal{P}^* \partial_t^2 t \mathcal{P} f\right)(x), \\ f(x) &= -2\left(\mathcal{P}^* \partial_t t \partial_t \mathcal{P} f\right)(x). \end{aligned}$$

Again, subtracting these two inversion formulas, we obtain the desired equality (2.34). The proof of Theorem A.4 is completed.

Remark D.3 *As shown in [21], identity (2.34) is the complete range description for the operator $\mathcal{P}$ (or, equivalently, for $\mathcal{T}$) when n is odd.*

2. The direct proof

Let u_1 be the extension of u by zero for $t < 0$, $\hat{u}_1$ be its Fourier transform, and $g_1 = u_1|_{S\times\mathbb{R}}$. Applying Lemma C.1, we have

$$\int_S \int_{\mathbb{R}} \overline{G}(|x-y|,\lambda)\hat{g}_0(y,\lambda)d\lambda d\sigma(y) = \int_S \mathcal{W}(g_0)(y,|x-y|)d\sigma(y),$$

and

$$\int_S \int_{\mathbb{R}} \overline{G}(|x-y|,\lambda)\hat{g}_1(y,\lambda)d\lambda d\sigma(y) = \int_S \mathcal{W}(g_1)(y,|x-y|)d\sigma(y).$$

Since $g_0 = g_1$ on $S\times[0,\infty)$, one has $\mathcal{W}(g_0)(y,|x-y|) = \mathcal{W}(g_1)(y,|x-y|)$. The above two equalities then give

$$\int_S \int_{\mathbb{R}} \overline{G}(|x-y|,\lambda)\hat{g}_0(y,\lambda)d\lambda d\sigma(y) = \int_S \int_{\mathbb{R}} \overline{G}(|x-y|,\lambda)\hat{g}_1(y,\lambda)d\lambda d\sigma(y). \tag{2.35}$$

Due to Proposition C.4,

$$\begin{aligned}
\hat{g}_1(y,\lambda) &= \frac{1}{2\pi}\int\limits_0^\infty g(y,s)e^{i\lambda s}ds = \frac{-i\lambda}{2\pi}\int\limits_0^\infty \mathcal{R}_S(f)(y,s)G(s,\lambda)ds \\
&= \frac{-i\lambda}{2\pi}\int\limits_{\mathbb{R}^n} f(z)G(|z-y|,\lambda)dz. \qquad (2.36)
\end{aligned}$$

From (2.35) and (2.36), we get

$$\begin{aligned}
&\int\limits_S\int\limits_{\mathbb{R}} \overline{G}(|x-y|,\lambda)\hat{g}_0(y,\lambda)d\lambda d\sigma(y) \\
= \;& \frac{-i}{2\pi}\int\limits_S\int\limits_{\mathbb{R}} \overline{G}(|x-y|,\lambda)\lambda\int\limits_{\mathbb{R}^n} G(|y-z|,\lambda)f(z)dzd\lambda d\sigma(y) \\
= \;& \frac{-i}{2\pi}\int\limits_B f(z)\left(\int\limits_{\mathbb{R}}\lambda\int\limits_S \overline{G}(|x-y|,\lambda)G(|y-z|,\lambda)d\sigma(y)d\lambda\right)dz.
\end{aligned}$$

That is,

$$\int\limits_S\int\limits_{\mathbb{R}} \overline{G}(|x-y|,\lambda)\hat{g}_0(y,\lambda)d\lambda d\sigma(y) = \frac{-i}{2\pi}\int\limits_B f(z)\left(\int\limits_{\mathbb{R}}\lambda\varphi(\lambda)d\lambda\right)dz, \qquad (2.37)$$

where

$$\varphi(\lambda) = \int\limits_S \overline{G}(|x-y|,\lambda)G(|y-z|,\lambda)d\sigma(y).$$

From the explicit formula for G in Proposition C.2, we see that $G(s,-\lambda) = \overline{G}(s,\lambda)$. Thus,

$$\varphi(-\lambda) = \overline{\varphi}(\lambda). \qquad (2.38)$$

Recall the notations $\mathcal{R}(s,\lambda) = Re(G(s,\lambda)), \mathcal{I}(s,\lambda) = Im(G(s,\lambda))$, and let

$$K(x,z,\lambda) = \int\limits_S \mathcal{R}(|x-y|,\lambda)\mathcal{I}(|z-y|,\lambda)d\sigma(y).$$

We claim the following symmetry whose proof is presented in Section F:

$$K(x,z,\lambda) = K(z,x,\lambda) \text{ for any } x,z\in B \text{ and } \lambda\in\mathbb{R}. \qquad (2.39)$$

Assuming this symmetry, one has $Im(\varphi(\lambda)) = K(x,z,\lambda) - K(z,x,\lambda) = 0$. From (2.38), we get $\varphi(-\lambda) = \varphi(\lambda)$. This implies the following equality:

$$\int\limits_{\mathbb{R}} \lambda\varphi(\lambda) d\lambda = 0.$$

Due to (2.37), one concludes that

$$\int\limits_{S}\int\limits_{\mathbb{R}} \overline{G}(|x-y|,\lambda)\hat{g}_0(y,\lambda) d\lambda d\sigma(y) = 0.$$

The proof is completed.

E. Some special cases

In this section, we derive Propositions A.6, A.7, A.9 and A.10 from Corollary A.5 by proper choices of ξ and φ.

1. Proposition A.6

By choosing $\xi = x$ and $\varphi(x) = c$ in (2.15), we obtain

$$f(x) = -2\int\limits_{S} \partial_s \mathcal{W}(g)(y,s)\big|_{s=|x-y|} |x-y| d\sigma(y) + c\int\limits_{S} \mathcal{W}(g)(y,|x-y|)d\sigma(y).$$

Here we use the notation ∂_s for $\frac{d}{ds}$. That is,

$$f(x) = -2\int\limits_{S} \left[s\partial_s \mathcal{W}g\right](y,|x-y|)d\sigma(y) + c\int\limits_{S} \mathcal{W}(g)(y,|x-y|)d\sigma(y). \tag{2.40}$$

We now claim an identity, whose proof can be found in Section F:

$$s\partial_s \mathcal{W}(g) = \mathcal{W}(s\partial_s g) - (n-2)\mathcal{W}(g). \tag{2.41}$$

Assuming this claim, we see that (2.40) is equivalent to

$$\begin{aligned} f(x) &= -2\int\limits_S \mathcal{W}(s\partial_s g)(y,|x-y|)d\sigma(y) \\ &+ [2(n-2)+c]\int\limits_S \mathcal{W}(g)(y,|x-y|)d\sigma(y). \end{aligned}$$

Due to Lemma D.2, we obtain

$$f(x) = -2\mathcal{P}^*\left(s\partial_s g\right)(x) + [2(n-2)+c]\mathcal{P}^*(g)(x). \tag{2.42}$$

- If $c = -2(n-2)$ then (2.42) becomes

$$f(x) = -2\mathcal{P}^*\left(s\partial_s g\right)(x) = -2(\mathcal{P}^* s\partial_s^2 \mathcal{P} f)(x)$$

- If $c = -2(n-1)$ then (2.42) becomes

$$f(x) = -2\mathcal{P}^*\left(s\partial_s g\right)(x) - 2\mathcal{P}^*(g)(x) = -2(\mathcal{P}^*\partial_s s\partial_s \mathcal{P} f)(x)$$

- If $c = -2n$ then (2.42) becomes

$$f(x) = -2\mathcal{P}^*\left(s\partial_s g\right)(x) - 4\mathcal{P}^*(g)(x) = -2(\mathcal{P}^*\partial_s^2 s\mathcal{P} f)(x).$$

Propositions A.6 is proved (in this proof we used the variable s in place of t).

2. Proposition A.7

Choosing $\varphi(x) = 0$ in (2.14), we obtain

$$f(x) = -\frac{4}{\pi}\int\limits_S \left(\frac{d}{ds}K_n(y,s)\right)_{s=|x-y|} \frac{\langle y-x, y-\xi\rangle}{|x-y|} d\sigma(y), \tag{2.43}$$

Due to (2.3) we have $G(s,\lambda) = \frac{\lambda^{n-2}}{4(2\pi)^{\frac{n-2}{2}}}\left[iJ(\lambda s) - N(\lambda s)\right]$. Thus,

$$\mathcal{R}(s,\lambda) = -\frac{\lambda^{n-2}}{4(2\pi)^{\frac{n-2}{2}}} N(\lambda s),\ \mathcal{I}(r,\lambda) = \frac{\lambda^{n-2}}{4(2\pi)^{\frac{n-2}{2}}} J(\lambda r). \tag{2.44}$$

Hence, for K_n defined in Theorem A.3,

$$\begin{aligned} K_n(y,s) &= \int_0^\infty \lambda \mathcal{R}(s,\lambda) \int_0^\infty \mathcal{R}_S(f)(y,r)\mathcal{I}(r,\lambda)dr d\lambda \\ &= \frac{-1}{16(2\pi)^{n-2}} \int_0^\infty \lambda^{2n-3} N(s\lambda) \int_0^\infty \mathcal{R}_S(f)(y,r)J(r\lambda)dr d\lambda. \end{aligned}$$

Since f is supported inside $\overline{B}$, we have $\mathcal{R}_S(f)(y,r) = 0$ for $r \geq 2$. Therefore,

$$\begin{aligned} K_n(y,s) &= \frac{-1}{16(2\pi)^{n-2}} \int_0^\infty \lambda^{2n-3} N(s\lambda) \int_0^2 \mathcal{R}_S(f)(y,r)J(r\lambda)dr d\lambda \\ &= \frac{-1}{16(2\pi)^{n-2}} k_n(y,s), \end{aligned}$$

where, as defined Proposition A.9,

$$k_n(y,s) = \int_0^\infty \lambda^{2n-3} N(s\lambda) \int_0^2 \mathcal{R}_S(y,r)J(r\lambda)dr d\lambda.$$

From (2.43), we arrive at

$$f(x) = \frac{1}{2(2\pi)^{n-1}} \int_S \left(\frac{d}{ds}k_n(y,s)\right)_{s=|x-y|} \frac{\langle y-x, y-\xi\rangle}{|x-y|} d\sigma(y), \tag{2.45}$$

Choosing $\xi = x$, we have

$$f(x) = \frac{1}{2(2\pi)^{n-1}} \int_S \left(\frac{d}{ds}k_n(y,s)\right)_{s=|x-y|} |x-y| d\sigma(y).$$

That is,

$$f(x) = \frac{1}{2(2\pi)^{n-1}} \int_S \left(s\frac{d}{ds}k_n(y,s)\right)_{s=|x-y|} d\sigma(y). \tag{2.46}$$

We claim that for even n

$$k_n(y,s) = \frac{(-1)^{\frac{n-2}{2}}}{\pi} \int_0^2 \left(\frac{d}{dr}\frac{1}{r}\right)^{n-1} \mathcal{R}_S(f)(y,r) \ln|r^2 - s^2| dr. \tag{2.47}$$

Here the integral is understood in the principal value sense. A proof of this claim will be given in Section 4. Assuming it, we obtain

$$\begin{aligned} s\frac{d}{ds}k_n(y,s) &= \frac{2(-1)^{\frac{n-2}{2}}}{\pi} \int_0^2 \left(\frac{d}{dr}\frac{1}{r}\right)^{n-1} \mathcal{R}_S(f)(y,r)\frac{s^2}{s^2-r^2}dr \\ &= \frac{2(-1)^{\frac{n-2}{2}}}{\pi} \int_0^2 \left(\frac{d}{dr}\frac{1}{r}\right)^{n-1} \mathcal{R}_S(f)(y,r)\left[1+\frac{r^2}{s^2-r^2}\right]dr. \end{aligned}$$

Since

$$\int_0^2 \left(\frac{d}{dr}\frac{1}{r}\right)^{n-1} \mathcal{R}_S(f)(y,r)dr = \frac{1}{r}\left(\frac{d}{dr}\frac{1}{r}\right)^{n-2} \mathcal{R}_S(f)(y,r)\Bigg|_0^2 = 0,$$

we get

$$\begin{aligned} s\frac{d}{ds}k_n(y,s) &= \frac{2(-1)^{\frac{n-2}{2}}}{\pi} \int_0^2 \left(\frac{d}{dr}\frac{1}{r}\right)^{n-1} \mathcal{R}_S(f)(y,r)\frac{r^2}{s^2-r^2}dr \\ &= \frac{2(-1)^{\frac{n}{2}}}{\pi} \int_0^2 r\left(\frac{d}{dr}\frac{1}{r}\right)^{n-1} \mathcal{R}_S(f)(y,r)\frac{r}{r^2-s^2}dr. \end{aligned}$$

Integrating by part (which can be justified for the principal value integral in question), we obtain

$$s\frac{d}{ds}k_n(y,s) = \frac{(-1)^{\frac{n-2}{2}}}{\pi} \int_0^2 \frac{d}{dr}r\left(\frac{d}{dr}\frac{1}{r}\right)^{n-1} \mathcal{R}_S(f)(y,r)\ln|r^2-s^2|dr.$$

From (2.46), we arrive at

$$f(x) = \frac{(-1)^{\frac{n-2}{2}}}{(2\pi)^n} \int_S \int_0^2 \left[\frac{d}{dr}r\left(\frac{d}{dr}\frac{1}{r}\right)^{n-1} \mathcal{R}_S(f)\right](y,r)\ln\left|r^2-|x-y|^2\right| dr d\sigma(y).$$

Equivalently, $f(x)$ is equal to

$$\frac{(-1)^{\frac{n-2}{2}}\omega_n}{(2\pi)^n}\int\limits_S\int\limits_0^2\left[\frac{d}{dr}r\left(\frac{d}{dr}\frac{1}{r}\right)^{n-1}r^{n-1}\mathcal{M}_S(f)\right](y,r)\ln\left|r^2-|x-y|^2\right|drd\sigma(y).$$

3. Propositions A.9 and A.10

Choosing $\xi = 0$ in (2.45), we obtain

$$f(x) = \frac{1}{2(2\pi)^{n-1}}\int\limits_S\left(\frac{d}{ds}k_n(y,s)\right)_{s=|x-y|}\frac{\langle y-x,y\rangle}{|x-y|}d\sigma(y),$$

That is,

$$f(x) = \frac{-1}{2(2\pi)^{n-1}}\nabla_x\cdot\int\limits_S n(y)k_n(y,|x-y|)d\sigma(y).$$

This proves Proposition A.9. In order to prove Proposition A.10, we need only show that $h(y,s) = 2k_n(y,s)$ for any $s > 0$. Indeed, it is a consequence of the following lemma whose proof can be found in Section F:

Lemma E.1 *Suppose that $h \in C^\infty[0,\infty)$ such that h does not grow too fast at infinity and $h^{(i)}(0) = 0$ for all $0 \le i \le n-3$ if n is odd, and $h^{(i)}(0) = 0$ for all $0 \le i \le n-2$ if n is even. Then*

$$\int\limits_0^\infty\lambda^{2n-3}N(s\lambda)\int\limits_0^\infty h(r)J(r\lambda)drd\lambda = -\int\limits_0^\infty\lambda^{2n-3}J(s\lambda)\int\limits_0^\infty h(r)N(r\lambda)d\lambda.$$

F. Proofs of auxiliary statements

1. Proof of Proposition C.2

We now prove

$$G(s,\lambda) = \begin{cases} c_n\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}}\left(\int\limits_s^\infty\frac{e^{i\lambda t}}{\sqrt{t^2-s^2}}dt\right), & \text{if } n \text{ is even,} \\ c_n\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-3}{2}}\left(\frac{e^{i\lambda s}}{s}\right), & \text{if } n \text{ is odd.}\end{cases} \tag{2.48}$$

Indeed, from (2.3), we get

$$G(s,\lambda) = \frac{i}{4}\left(\frac{\lambda}{2\pi s}\right)^{\frac{n-2}{2}} H^{(1)}_{\frac{n-2}{2}}(\lambda s) = \frac{i}{4}\frac{\lambda^{n-2}}{(2\pi)^{\frac{n-2}{2}}}\frac{H^{(1)}_{\frac{n-2}{2}}(\lambda s)}{(\lambda s)^{\frac{n-2}{2}}}, \tag{2.49}$$

where $H^{(1)}_{\frac{n-2}{2}}$ is the Hankel function of the first kind.

- Let n be an even number. Using the equality (e.g. [63, page 74])

$$\frac{H^{(1)}_{\nu+m}(s)}{s^{\nu+m}} = (-1)^m \left(\frac{1}{s}\frac{d}{ds}\right)^m \left(\frac{H^{(1)}_{\nu}(s)}{s^{\nu}}\right), \tag{2.50}$$

we obtain

$$\begin{aligned}\frac{H^{(1)}_{\frac{n-2}{2}}(\lambda s)}{(\lambda s)^{\frac{n-2}{2}}} &= (-1)^{\frac{n-2}{2}} \left[\left(\frac{1}{t}\frac{d}{dt}\right)^{\frac{n-2}{2}} H^{(1)}_0(t)\right]\Bigg|_{t=\lambda s} \\ &= (-1)^{\frac{n-2}{2}}\lambda^{-(n-2)}\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}} H^{(1)}_0(\lambda s).\end{aligned}$$

Hence, (2.49) gives

$$G(s,\lambda) = \frac{i}{4}\frac{(-1)^{\frac{n-2}{2}}}{(2\pi)^{\frac{n-2}{2}}}\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}} H^{(1)}_0(\lambda s).$$

Since (e.g., [63, page 170])

$$H^{(1)}_0(s) = \frac{-2i}{\pi}\int_1^{\infty}\frac{e^{ist}}{\sqrt{t^2-1}}dt = \frac{-2i}{\pi}\int_s^{\infty}\frac{e^{it}}{\sqrt{t^2-s^2}}dt,$$

we conclude that

$$G(s,\lambda) = \frac{(-1)^{\frac{n-2}{2}}}{(2\pi)^{\frac{n}{2}}}\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}}\left(\int_s^{\infty}\frac{e^{i\lambda t}}{\sqrt{t^2-s^2}}dt\right).$$

This confirms (2.48) for even n.

- Let n be an odd number. Using (2.50) again, one gets

$$\frac{H^{(1)}_{\frac{n-2}{2}}(\lambda s)}{(\lambda s)^{\frac{n-2}{2}}} = (-1)^{\frac{n-3}{2}}\lambda^{-(n-3)}\left[\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-3}{2}}\frac{H^{(1)}_{\frac{1}{2}}(\lambda s)}{(\lambda s)^{\frac{1}{2}}}\right].$$

Hence, from (2.49), we obtain

$$G(s,\lambda) \quad = \quad \frac{i}{4}\frac{(-1)^{\frac{n-3}{2}}\lambda}{(2\pi)^{\frac{n-2}{2}}}\left[\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-3}{2}}\frac{H^{(1)}_{\frac{1}{2}}(\lambda s)}{(\lambda s)^{\frac{1}{2}}}\right].$$

Since (e.g., [10, page 487])

$$H^{(1)}_{\frac{1}{2}}(s) = -i\sqrt{\frac{2}{\pi s}}e^{is},$$

we conclude that

$$G(s,\lambda) = \frac{(-1)^{\frac{n-3}{2}}}{2(2\pi)^{\frac{n-1}{2}}}\left[\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-3}{2}}\frac{e^{i\lambda s}}{s}\right].$$

This confirms (2.48) for odd n.

2. Proof of identity (2.39)

We prove that for all $\lambda \in \mathbb{R}$, the function

$$K(x,z,\lambda) = \int\limits_S \mathcal{R}(|x-y|,\lambda)\mathcal{I}(|z-y|,\lambda)d\sigma(y),$$

is symmetric with respect to $x, z \in B$. We now follow the line of reasoning used in [38].

Due to (2.44), we get

$$K(x,z,\lambda) = c_0\int\limits_S N(\lambda|x-y|)J(\lambda|z-y|)d\sigma(y), \tag{2.51}$$

where c_0 is a constant depending only on n and λ. We recall the following identities from [38] for $|x| > r_0$ (c_1 depends only on n and λ):

$$\int\limits_{|y|=r_0} Y^k_l(\hat{y})J(\lambda|x-y|)dy \quad = \quad c_1 r_0^{n-1}J_{(k)}(\lambda r_0)J_{(k)}(\lambda|x|)Y^k_l(\hat{x}), \tag{2.52}$$

$$\int\limits_{|y|=r_0} Y_l^k(\hat{y})N(\lambda|x-y|)dy \quad = \quad c_1 r_0^{n-1} J_{(k)}(\lambda r_0) N_{(k)}(\lambda|x|) Y_l^k(\hat{x}). \tag{2.53}$$

Here, $\hat{x} = \frac{x}{|x|}$, Y_l^k is a spherical harmonic of order k, and

$$J_{(k)}(t) = \frac{J_{\frac{n-2}{2}+k}(t)}{t^{\frac{n-2}{2}}}, \quad N_{(k)}(t) = \frac{N_{\frac{n-2}{2}+k}(t)}{t^{\frac{n-2}{2}}}.$$

Consider the spherical harmonic expansion

$$K(x,z,\lambda) = \sum_{(k,l),(k',l')} a_{k',l'}^{k,l}(\alpha,\beta) Y_l^k(\hat{x}) Y_{l'}^{k'}(\hat{z}),$$

where $\alpha = |x|$ and $\beta = |z|$.

Due to (2.51), we obtain

$$\begin{aligned}
& a_{k',l'}^{k,l}(\alpha,\beta) \\
&= \int\limits_S \int\limits_S Y_l^k(\hat{x}) Y_{l'}^{k'}(\hat{z}) K(\alpha\hat{x}, \beta\hat{z}, \lambda) d\sigma(\hat{x}) d\sigma(\hat{z}) \\
&= c_0 \int\limits_S \int\limits_S Y_l^k(\hat{x}) Y_{l'}^{k'}(\hat{z}) \int\limits_S N(\lambda|y-\alpha\hat{x}|) J(\lambda|y-\beta\hat{z}|) d\sigma(y) d\hat{x} d\hat{z} \\
&= c_0 \int\limits_S \left(\int\limits_S Y_l^k(\hat{x}) N(\lambda|y-\alpha\hat{x}|) d\hat{x} \right) \left(\int\limits_S Y_{l'}^{k'}(\hat{z}) J(\lambda|y-\beta\hat{z}|) d\hat{z} \right) d\sigma(y).
\end{aligned}$$

Applying (2.52) and (2.53), we arrive at

$$\begin{aligned}
a_{k',l'}^{k,l}(\alpha,\beta) &= c_0 c_1^2 J_{(k)}(\lambda\alpha) J_{(k')}(\lambda\beta) N_{(k)}(\lambda) J_{(k')}(\lambda) \int\limits_S Y_k^l(y) Y_{k'}^{l'}(y) d\sigma(y) \\
&= c_0 c_1^2 J_{(k)}(\lambda\alpha) J_{(k')}(\lambda\beta) N_{(k)}(\lambda) J_{(k')}(\lambda) \delta_{k,k'} \delta_{l,l'}.
\end{aligned}$$

Hence, $a_{k,l}^{k,l}(\alpha,\beta) = a_{k,l}^{k,l}(\beta,\alpha)$ and $K(x,z,\lambda) = \sum_{(k,l)} a_{k,l}^{k,l}(\alpha,\beta) Y_l^k(\hat{x}) Y_l^k(\hat{z})$. These two equalities give $K(x,z,\lambda) = K(z,x,\lambda)$. The proof is completed.

3. Proof of identity (2.41)

We now prove the identity

$$\left[s\frac{d}{ds}\mathcal{W}\right](g) = \left[\mathcal{W}s\frac{d}{ds}\right](g) - (n-2)\mathcal{W}(g). \tag{2.54}$$

We first recall from (2.7)

$$\mathcal{W}(g)(y,s) := \begin{cases} c_n \left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}} \int\limits_s^\infty \frac{g(y,t)}{\sqrt{t^2-s^2}}dt, \text{ if } n \text{ is even,} \\ c_n \left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-3}{2}} \left(\frac{g(y,s)}{s}\right), \text{ if } n \text{ is odd.} \end{cases} \tag{2.55}$$

- For $n = 2$, due to (2.55), we have

$$\left(s\frac{d}{ds}\mathcal{W}\right)(g)(y,s) = c_2\left(s\frac{d}{ds}\right)\int\limits_s^\infty \frac{g(y,t)}{\sqrt{t^2-s^2}}dt.$$

Taking integration by parts, one gets

$$\int\limits_s^\infty \frac{g(y,t)}{\sqrt{t^2-s^2}}dt \quad = \quad -\int\limits_s^\infty \frac{d}{dt}\left(\frac{g(y,t)}{t}\right)\sqrt{t^2-s^2}dt.$$

Hence,

$$\begin{aligned} s\frac{d}{ds}\int\limits_s^\infty \frac{g(y,t)}{\sqrt{t^2-s^2}}dt &= -s\int\limits_s^\infty \frac{d}{dt}\left(\frac{g(y,t)}{t}\right)\frac{d}{ds}\left(\sqrt{t^2-s^2}\right)dt \\ &= \int\limits_s^\infty \frac{d}{dt}\left(\frac{g(y,t)}{t}\right)\frac{s^2}{\sqrt{t^2-s^2}}dt \\ &= \int\limits_s^\infty \frac{d}{dt}\left(\frac{g(y,t)}{t}\right)\frac{t^2}{\sqrt{t^2-s^2}}dt - \int\limits_s^\infty \frac{d}{dt}\left(\frac{g(y,t)}{t}\right)\sqrt{t^2-s^2}dt \end{aligned}$$

Taking integration by parts again, we arrive at

$$\begin{aligned} s\frac{d}{ds}\int\limits_s^\infty \frac{g(y,t)}{\sqrt{t^2-s^2}}dt \quad &= \quad \int\limits_s^\infty \frac{d}{dt}\left(\frac{g(y,t)}{t}\right)\frac{t^2}{\sqrt{t^2-s^2}}dt \\ &+ \quad \int\limits_s^\infty \left(\frac{g(y,t)}{t}\right)\frac{t}{\sqrt{t^2-s^2}}dt. \end{aligned}$$

Simplifying the right hand side, we get

$$s\frac{d}{ds}\int_s^\infty \frac{g(y,t)}{\sqrt{t^2-s^2}}dt = \int_s^\infty \frac{tg_t(y,t)}{\sqrt{t^2-s^2}}dt. \tag{2.56}$$

Here $g_t(y,t)$ is the derivative of g with respect to t. Therefore,

$$\left(s\frac{d}{ds}\mathcal{W}\right)(g)(y,s) = c_2\int_s^\infty \frac{tg_t(y,t)}{\sqrt{t^2-s^2}}dt = \left(\mathcal{W}s\frac{d}{ds}\right)g(y,s).$$

This confirms (2.54) for $n=2$.

- Let $n>2$ be even. We first observe the simple identity

$$\left(s\frac{d}{ds}\right)\left(\frac{1}{s}\frac{d}{ds}\right) = \left(\frac{1}{s}\frac{d}{ds}\right)\left(s\frac{d}{ds}\right) - 2\left(\frac{1}{s}\frac{d}{ds}\right).$$

By induction, we get

$$\left(s\frac{d}{ds}\right)\left(\frac{1}{s}\frac{d}{ds}\right)^k = \left(\frac{1}{s}\frac{d}{ds}\right)^k\left(s\frac{d}{ds}\right) - 2k\left(\frac{1}{s}\frac{d}{ds}\right)^k. \tag{2.57}$$

Therefore, due to the definition of $\mathcal{W}$ in (2.55),

$$\begin{aligned}\left(s\frac{d}{ds}\mathcal{W}\right)(g) &= c_n\left(s\frac{d}{ds}\right)\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}}\int_s^\infty \frac{g(y,t)}{\sqrt{t^2-s^2}}dt\\ &= c_n\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}}\left(s\frac{d}{ds}\right)\int_s^\infty \frac{g(y,t)}{\sqrt{t^2-s^2}}dt\\ &- (n-2)c_n\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}}\int_s^\infty \frac{g(y,t)}{\sqrt{t^2-s^2}}dt.\end{aligned}$$

Due to (2.56), we obtain

$$\begin{aligned}\left(s\frac{d}{ds}\right)\mathcal{W}(g)(y,s) &= c_n\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}}\int_s^\infty \frac{tg_t(y,t)}{\sqrt{t^2-s^2}}dt\\ &- (n-2)c_n\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}}\int_s^\infty \frac{g(y,t)}{\sqrt{t^2-s^2}}dt\end{aligned}$$

$$= \left(\mathcal{W}s\frac{d}{ds}\right)(g)(y,s) - (n-2)\mathcal{W}(g)(y,s).$$

This confirms (2.54) for any even n.

- Let n be odd. Due to (2.55),

$$\left(s\frac{d}{ds}\mathcal{W}\right)(g)(y,s) = c_n\left(s\frac{d}{ds}\right)\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-3}{2}}\left(\frac{g(y,s)}{s}\right).$$

This and (2.57) give

$$\begin{aligned}\left(s\frac{d}{ds}\mathcal{W}\right)(g)(y,s) &= c_n\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-3}{2}}\left(s\frac{d}{ds}\right)\left(\frac{g(y,s)}{s}\right)\\ &- c_n(n-3)\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-3}{2}}\left(\frac{g(y,s)}{s}\right).\end{aligned}$$

Using the identity $\left(s\frac{d}{ds}\right)\left(\frac{g(y,s)}{s}\right) = g_s(y,s) - \frac{g(y,s)}{s}$ for the first term of the right hand side of the above equality, we obtain

$$\begin{aligned}\left(s\frac{d}{ds}\mathcal{W}\right)(g)(y,s) &= c_n\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-3}{2}} g_s(y,s)\\ &- c_n(n-2)\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-3}{2}}\left(\frac{g(y,s)}{s}\right)\\ &= \left(\mathcal{W}s\frac{d}{ds}\right)(g)(y,s) - (n-2)\mathcal{W}(g)(y,s).\end{aligned}$$

This finishes the proof of identity (2.54).

4. Proof of (2.47)

We prove a more general result:

Lemma F.1 *Assume that n is even. Let $g \in C_0^\infty(S \times [0,\infty))$ and*

$$k_n(y,s) = \int_0^\infty \lambda^{2n-3} N(s\lambda) \int_0^\infty g(y,r) J(r\lambda)\,dr\,d\lambda.$$

Then

$$k_n(y,s)=\frac{(-1)^{\frac{n-2}{2}}}{\pi}\int\limits_0^\infty\left(\frac{d}{dr}\frac{1}{r}\right)^{n-1}g(y,r)\ln|r^2-s^2|dr.$$

Obviously, applying this lemma for $g(y,s)=(\mathcal{R}_S f)(y,s)$ we obtain (2.47).

Proof Since n is even, due to (2.50), we have

$$N(s)=(-1)^{\frac{n-2}{2}}\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}}N_0(s),\ \ J(s)=(-1)^{\frac{n-2}{2}}\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}}J_0(s).$$

Thus,

$$k_n(y,s)=\int\limits_0^\infty\lambda^{2n-3}\left[\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}}N_0\right](s\lambda)\int\limits_0^\infty g(y,r)\left[\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}}J_0\right](r\lambda)drd\lambda.$$

Since

$$\left[\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}}N_0\right](s\lambda)=\lambda^{-(n-2)}\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}}N_0(s\lambda),$$

and

$$\left[\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}}J_0\right](s\lambda)=\lambda^{-(n-2)}\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}}J_0(s\lambda),$$

we have

$$\begin{aligned} k_n(y,s) &= \int\limits_0^\infty\lambda\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}}N_0(s\lambda)\int\limits_0^\infty g(y,r)\left(\frac{1}{r}\frac{d}{dr}\right)^{\frac{n-2}{2}}J_0(r\lambda)drd\lambda \\ &= \left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}}\int\limits_0^\infty\lambda N_0(s\lambda)\int\limits_0^\infty g(y,r)\left(\frac{1}{r}\frac{d}{dr}\right)^{\frac{n-2}{2}}J_0(r\lambda)drd\lambda. \end{aligned}$$

Integrating by parts for the inner integral, we obtain

$$k_n(y,s)=(-1)^{\frac{n-2}{2}}\left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-2}{2}}l_n(y,s), \tag{2.58}$$

where

$$l_n(y,s)=\int\limits_0^\infty\lambda N_0(s\lambda)\int\limits_0^\infty\left(\frac{d}{dr}\frac{1}{r}\right)^{\frac{n-2}{2}}g(y,r)J_0(r\lambda)drd\lambda.$$

Following the argument in [3], we obtain

$$l_n(y,s) \quad = \quad -\frac{2}{\pi}\int\limits_0^\infty \left(\frac{d}{dr}\frac{1}{r}\right)^{\frac{n-2}{2}} g(y,r)\frac{1}{r^2-s^2}dr.$$

Here the integral is understood in the principal value sense. We now prove that

$$\left(\frac{1}{s}\frac{d}{ds}\right)^\alpha l_n(y,s) = -\frac{2}{\pi}\int\limits_0^\infty \left(\frac{d}{dr}\frac{1}{r}\right)^{\frac{n-2}{2}+\alpha} g(y,r)\frac{1}{r^2-s^2}dr,$$

for any nonnegative integer α. Indeed, we can rewrite

$$l_n(y,s) \quad = \quad -\frac{2}{\pi}\int\limits_0^\infty \frac{1}{r}\left(\frac{d}{dr}\frac{1}{r}\right)^{\frac{n-2}{2}} g(y,r)\frac{r}{r^2-s^2}dr.$$

Integrating by parts (which can be justified for the principal value integral in question), we get

$$l_n(y,s) \quad = \quad \frac{1}{\pi}\int\limits_0^\infty \frac{d}{dr}\frac{1}{r}\left(\frac{d}{dr}\frac{1}{r}\right)^{\frac{n-2}{2}} g(y,r)\ln|r^2-s^2|dr.$$

Hence,

$$\begin{aligned}\left(\frac{1}{s}\frac{d}{ds}\right) l_n(y,s) \quad &= \quad \frac{2}{\pi}\int\limits_0^\infty \frac{d}{dr}\frac{1}{r}\left(\frac{d}{dr}\frac{1}{r}\right)^{\frac{n-2}{2}} g(y,r)\frac{1}{s^2-r^2}dr \\ &= \quad -\frac{2}{\pi}\int\limits_0^\infty \left(\frac{d}{dr}\frac{1}{r}\right)^{\frac{n-2}{2}+1} g(y,r)\frac{1}{r^2-s^2}dr.\end{aligned}$$

By induction on α, we conclude that

$$\left(\frac{1}{s}\frac{d}{ds}\right)^\alpha l_n(y,s) \quad = \quad -\frac{2}{\pi}\int\limits_0^\infty \left(\frac{d}{dr}\frac{1}{r}\right)^{\frac{n-2}{2}+\alpha} g(y,r)\frac{1}{r^2-s^2}dr,$$

for any nonnegative integer α. Therefore, due to (2.58),

$$k_n(y,s) = \frac{2(-1)^{\frac{n}{2}}}{\pi}\int\limits_0^\infty \left(\frac{d}{dr}\frac{1}{r}\right)^{n-2} g(y,r)\frac{1}{r^2-s^2}dr$$

Integrating by parts again, we obtain

$$k_n(y,s) = \frac{(-1)^{\frac{n-2}{2}}}{\pi} \int_0^\infty \left(\frac{d}{dr}\frac{1}{r}\right)^{n-1} g(y,r)\ln|r^2 - s^2|dr.$$

∎

5. Proof of Lemma E.1

In view of (2.44), we have

$$\begin{aligned}
\int_0^\infty \lambda^{2n-3} N(s\lambda) \int_0^\infty h(r)J(r\lambda)drd\lambda &= c\int_0^\infty \lambda\mathcal{R}(s,\lambda)\int_0^\infty h(r)\mathcal{I}(r,\lambda)drd\lambda \quad (2.59)\\
\int_0^\infty \lambda^{2n-3} J(s\lambda) \int_0^\infty h(r)N(r\lambda)drd\lambda &= c\int_0^\infty \lambda\mathcal{I}(s,\lambda)\int_0^\infty h(r)\mathcal{R}(r,\lambda)drd\lambda. \quad (2.60)
\end{aligned}$$

Here the constant c is the same in these two formulas. Recall that,

$$G(s,\lambda) = \mathcal{W}(e_\lambda)(s),$$

where $e_\lambda(s) = e^{i\lambda s}$. Let $\cos_\lambda(s) = \cos(\lambda s)$ and $\sin_\lambda(s) = \sin(\lambda s)$, we then have

$$\int_0^\infty \lambda\mathcal{R}(s,\lambda)\int_0^\infty h(r)\mathcal{I}(r,\lambda)drd\lambda = \int_0^\infty \lambda\mathcal{W}(\cos_\lambda)(s)\int_0^\infty h(r)\mathcal{W}(\sin_\lambda)(r)drd\lambda.$$

Similar to the argument in Lemma C.1, we can take $\mathcal{W}$ out of the integral sign to obtain

$$\int_0^\infty \lambda\mathcal{R}(s,\lambda)\int_0^\infty h(r)\mathcal{I}(r,\lambda)drd\lambda = \mathcal{W}(H)(s), \quad (2.61)$$

where

$$\begin{aligned}
H(s) &= \int_0^\infty \lambda\cos(\lambda s)\int_0^\infty h(r)\mathcal{W}(\sin_\lambda)drd\lambda\\
&= \frac{d}{ds}\int_0^\infty \sin(\lambda s)\int_0^\infty h(r)\mathcal{W}(\sin_\lambda)drd\lambda.
\end{aligned}$$

Let $\mathcal{W}^*$ be the L^2-adjoint of $\mathcal{W}$, which is

$$\mathcal{W}^*(v)(s) = \begin{cases} (-1)^{\frac{n-2}{2}} c_n \int\limits_0^s \frac{\left(\frac{1}{t}\frac{d}{dt}\right)^{\frac{n-2}{2}} v(t)}{\sqrt{s^2-t^2}} dt, \text{ if } n \text{ is even,} \\ (-1)^{\frac{n-3}{2}} c_n \left(\frac{1}{s}\frac{d}{ds}\right)^{\frac{n-3}{2}} \left(\frac{v(s)}{s}\right), \text{ if } n \text{ is odd.} \end{cases} \tag{2.62}$$

Given the condition in Lemma E.1, integrating by parts (and changing the order of integration if n is even), we obtain

$$\int\limits_0^\infty h(r)\mathcal{W}(\sin_\lambda)drd\lambda = \int\limits_0^\infty \mathcal{W}^*(h)(r)\sin_\lambda(r)drd\lambda.$$

Therefore,

$$H(s) = \frac{d}{ds}\int\limits_0^\infty \sin(\lambda s)\int\limits_0^\infty \mathcal{W}^*(h)(r)\sin(\lambda r)drd\lambda.$$

Since the Fourier-sine transform inverts itself, we arrive at

$$H(s) = \frac{d}{ds}\mathcal{W}^*(h)(s).$$

Thus, due to (2.61),

$$\int\limits_0^\infty \lambda\mathcal{R}(s,\lambda)\int\limits_0^\infty h(r)J(r,\lambda)drd\lambda = \left(\mathcal{W}\frac{d}{ds}\mathcal{W}^*\right)(h)(s). \tag{2.63}$$

Similarly, since

$$\int\limits_0^\infty \lambda\mathcal{I}(s,\lambda)\int\limits_0^\infty h(r)\mathcal{R}(r,\lambda)drd\lambda = \int\limits_0^\infty \lambda\mathcal{W}(\sin_\lambda)(s)\int\limits_0^\infty h(r)\mathcal{W}(\cos_\lambda)(r)drd\lambda,$$

we obtain

$$\int\limits_0^\infty \lambda\mathcal{I}(s,\lambda)\int\limits_0^\infty h(r)\mathcal{R}(r,\lambda)drd\lambda = \mathcal{W}(H_1)(s),$$

where

$$H_1(s) = \int\limits_0^\infty \lambda\sin(\lambda s)\int\limits_0^\infty \mathcal{W}^*(h)(r)\cos(\lambda r)drd\lambda$$

$$= -\frac{d}{ds}\int_0^\infty \cos(\lambda s)\int_0^\infty \mathcal{W}^*(h)(r)\cos(\lambda r)drd\lambda$$
$$= -\frac{d}{ds}\mathcal{W}^*(h)(s),$$

Here, we have used the fact that the Fourier-cosine transform inverts itself. Thus,

$$\int_0^\infty \lambda\mathcal{R}(s,\lambda)\int_0^\infty h(r)J(r,\lambda)drd\lambda = -\left(\mathcal{W}\frac{d}{ds}\mathcal{W}^*\right)(h)(s). \tag{2.64}$$

From (2.59), (2.60), (2.63), and (2.64), we have

$$\int_0^\infty \lambda^{2n-3}N(s\lambda)\int_0^\infty h(r)J(r\lambda)drd\lambda = -\int_0^\infty \lambda^{2n-3}J(s\lambda)\int_0^\infty h(r)N(r\lambda)drd\lambda.$$

Lemma E.1 is proved.

G. Remarks

- The operator $\mathcal{W}$ is an intertwining operator between the second derivative and Bessel operator (e.g., [4, 40, 59]), i.e.

$$\mathcal{W}\left[\left(\frac{d}{ds}\right)^2 v\right] = \left[\left(\frac{d}{ds}\right)^2 + \frac{n-1}{s}\frac{d}{ds}\right]\mathcal{W}(v).$$

  Thus, $\mathcal{W}$ transforms the wave equation into the Darboux equation, which is known to describe spherical means (see [7, 11, 28, 34]).

- The symmetry of $K(x,z,\lambda)$ is similar to that of $I(x,z,\lambda)$ in [38]. It implies Theorem A.4 as shown in Section 2. It is not clear that the converse is true. However, a close look at the argument in Section 2 shows that Theorem A.4 implies the following weaker symmetry:

$$\int_{\mathbb{R}} K(x,z,\lambda)d\lambda = \int_{\mathbb{R}} K(z,x,\lambda)d\lambda, \text{ for all } x,z\in B.$$

- In [22], the authors derived an inversion formula using the known Neumann rather than Dirichlet observation data of the solution for wave solution for $n = 3$ (see [22, Theorem 12]). Applying the results in this paper, one can derive the inversion formulas from Neumann data for any n. Indeed, looking at (2.25) and (2.26), we arrive at:

$$f(x) = 2 \int_S \int_{\mathbb{R}} \overline{G}(|y-x|, \lambda) \frac{\partial \hat{u}_0}{\partial \nu_y}(y, \lambda) d\lambda d\sigma(y).$$

  Due to Lemma C.1, we obtain

$$f(x) = 2 \int_S \mathcal{W}(\partial_{\nu_y} u)(y, |x-y|) d\sigma(y),$$

  where ∂_{ν_y} stands for the (outward pointing) normal derivative.

CHAPTER III

RANGE DESCRIPTION [1]

In this chapter, we describe the range of the spherical mean Radon transform $\mathcal{M}_S$, which evaluates mean values of a function on $\mathbb{R}^n$ over all spheres centered on the unit sphere S. This transform has important applications in TAT in acoustically homogeneous medium. Range description for $\mathcal{M}_S$ has been obtained recently. It includes smoothness and support condition, orthogonality condition and, for even dimensions, moment condition. However, it was found out later that, in any dimensions, the moment condition follows from the other ones, and therefore can be dropped. In terms of Darboux equation, which describes spherical means, this implies that solutions of certain boundary value problems in the unit ball B automatically extend to $\mathbb{R}^n$. We present here a direct proof of this global extendibility phenomenon for Darboux equation. Correspondingly, it delivers an alternative proof of the range characterization.

A. Introduction

Let us recall the definition of the spherical mean Radon transform:

$$\mathcal{M}(f)(x,t) := \frac{1}{\omega_n} \int\limits_{S^{n-1}} f(x+t\theta)dA(\theta),$$

where $dA(\theta)$ is the area measure on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ and ω_n is the total measure of the unit sphere. We now consider the model of TAT in acoustically

---

[1] Reprinted with permission from "Range conditions for a spherical mean transform and global extendibility of solutions of the Darboux equation", Mark Agranovsky and Linh V. Nguyen, accepted for publication in Journal d'Analyse Mathématique,

homogeneous medium:

$$\begin{cases} u_{tt}(x,t) - \triangle u(x,t) = 0, \ x \in \mathbb{R}^n, \ t \geq 0, \\ u(x,0) = f(x), \quad u_t(x,0) = 0, \\ u(y,t) = g(y,t), \ \text{for } y \in S, \ t \geq 0. \end{cases} \tag{3.1}$$

The standard Kirchhoff-Poisson solution formulas for the wave equation (see[11] or [15, p.77]) imply the representation

$$u(x,t) = c\left[\left(\frac{\partial}{\partial t}\frac{1}{t}\right)^{\frac{n-1}{2}} t^{n-1}\mathcal{M}(f)\right](x,t), \quad x \in \mathbb{R}^n, \quad t > 0. \tag{3.2}$$

Due to this relation, $\mathcal{M}$ is of very much interest of mathematics of TAT. Moreover, this transform actually has been intensively investigated in the literature due to its applications in PDEs and geophysics (e.g., [34, 11, 8, 9, 14, 24, 3]).

It is well known that $G(x,t) = \mathcal{M}(f)(x,t)$ satisfies the following problem for the Darboux equation [11, 34]:

$$\begin{cases} \mathcal{D}G(x,t) = 0, \ (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, \\ G(x,0) = f(x), \ G_t(x,0) = 0. \end{cases} \tag{3.3}$$

Here, $\mathbb{R}_+ = (0,\infty)$ and $\mathcal{D}$ is the Darboux operator:

$$\mathcal{D} = \partial_t^2 + \frac{n-1}{t}\partial_t - \Delta_x. \tag{3.4}$$

Moreover, Aisgeirsson's theorem [11, Ch.6] states that any global C^2 solution $G(x,t)$ of (3.3) is the spherical means of its initial value: $G(x,t) = \mathcal{M}(f)(x,t)$.

Since the data $g(y,t)$ of TAT are measured only on the observation surface S, we are interested in the restriction $\mathcal{M}_S$ of $\mathcal{M}$, which sends the function f to its mean values on the spheres centered on S. In additions to TAT, the problem of recovering functions from their spherical means restricted to such surfaces has interesting applications in analysis, in particular, in approximation theory (see [5, 41]).

In this chapter, we are concerned with the problem of characterizing functions h that belong to the range of $\mathcal{M}_S$.

Let us describe the problem in more precise terms (for the detailed exposition, we refer the reader to the articles [6, 21, 4]). First of all, in our considerations, the hypersurface S will be the unit sphere centered at the origin. We assume that $f \in C^\infty(\mathbb{R}^n)$ is supported inside the closure $\overline{B}$ of the unit ball $B = \{|x| < 1\}$ (the class of such functions f will be denoted by $C_0^\infty(\overline{B})$).

We consider the problem of characterizing all functions $h(x,t)$ on the cylinder $\Gamma = S \times \mathbb{R}_+$ such that $h = \mathcal{M}_S(f)$ for some function $f \in C_0^\infty(\overline{B})$.

Some necessary conditions are almost obvious. Firstly, the function $h(x,t) = \mathcal{M}_S(f)(x,t)$ must be smooth on Γ. Secondly, $h(x,t)$ must vanish for all $t > 2$ and also vanish to infinite order at $t = 0$. Thus, h must satisfy *smoothness and support condition*: $h \in C_0^\infty(S \times [0,2])$.

Another, less trivial, necessary condition can be derived from characterization of the spherical means by Darboux equation (3.3). This condition, which we call *orthogonality condition* (see Theorem C.2), is of Fredholm alternative type. It follows from the existence of a solution of Darboux equation inside the cylinder Γ (the fact that S is a sphere is not important here).

The first complete range description for $n = 2$ was obtained in [6]. It was proved that a function h belongs to the image of $C_0^\infty(\overline{B})$ under the operator $\mathcal{M}_S$ if and only if it satisfies, besides the smoothness and support condition and the above orthogonality condition, an additional *moment condition*. Earlier, the necessity of the moment condition was observed in [49].

Further step in higher dimensions was taken in [21]. There a range characterization was obtained for odd dimensions and for a transform related to the wave equation, rather than the Darboux equation. The characterization in [21] did not involve the moment condition. A complete description of the range for the spherical mean transform $\mathcal{R}_S$ in any dimension was obtained in [4]. As in [6], the necessary and sufficient conditions in [4] include: smoothness and support, orthogonality, and moment conditions, although the moment conditions were needed for even dimensions only. It is worth mentioning that the conditions of moment type but in a stronger form still can characterize the range, even without the orthogonality condition in [4] (see, e.g., [48]).

It had remained unclear whether the moment condition is really needed for even n, till the recent article [2]. It was proved there that, regardless of parity of the dimension, the moment condition follows from smoothness and support, and orthogonality conditions. Therefore, it can be completely dropped from the characterization.

The above result can be immediately translated to the language of Darboux equation. Namely, on one hand, it was proved in [4] that the orthogonality condition for the data $h(x,t)$ is in fact the condition of existence of a solution H of Darboux equation (3.3) inside the cylinder Γ with the boundary data $H(x,t) = h(x,t)$ on Γ and proper decay when $t \to +\infty$. On the other hand, spherical means are global solutions of Darboux equation. Therefore, h is in the range of transform $\mathcal{M}_S$ means that H extends as a global solution to the entire space $\mathbb{R}^n$.

The possibility of such an extension seems to be an interesting observation by itself and one may wish to have its direct proof. In fact, such a proof was found in [4], but only in odd dimensions. In this chapter, we modify the construction of [4] to extend it to all dimensions. Correspondingly, we obtain an alternative proof, universal for all dimensions, of the range characterization theorem from [2].

B. Main result

Let us recall here that B is the unit ball centered at the origin, $S = \partial B$ is the unit sphere, and $\Gamma = S \times \mathbb{R}_+$. We also denote by J_μ the Bessel function of the first kind of order μ and $j_\mu(u) = u^{-\mu} J_\mu(u)$ the corresponding normalized Bessel function. The notation $C_0^\infty(\overline{B})$ will stand for the space of smooth functions in $\mathbb{R}^n$ supported inside $\overline{B}$.

1. Formulation of main result

The goal of this chapter is to present a proof of the following result from [2]:

Theorem B.1 *Let h be a function defined on the cylinder Γ. Then there exists $f \in C_0^\infty(\overline{B})$ such that $h = \mathcal{M}_S(f)$ if and only if the following conditions hold:*

a) **Smoothness and support condition:** *$h \in C_0^\infty(\Gamma)$ and $h(x,t) = 0$ when $t > 2$.*

b) **Orthogonality condition:** *Let $-\lambda^2$ be an eigenvalue of the Dirichlet Laplacian on B and φ_λ a corresponding eigenfunction. Then*

$$\int\limits_\Gamma h(x,t)\partial_{\nu_x}\varphi_\lambda(x) j_{\frac{n-2}{2}}(\lambda t) t^{n-1} dt dA(x) = 0,$$

where ν_x is the outward normal to S at x.

Remark: Since S is the unit sphere, the Dirichlet eigenfunctions φ_λ can be written in polar coordinates as follows:

$$\varphi_\lambda(r\theta) = j_{\frac{n+m-2}{2}}(\lambda r) Y_m(\theta),$$

where Y_m is a spherical harmonics of degree m. The Dirichlet condition for φ_λ on the unit sphere S requires that $j_{(n+m-2)/2}(\lambda) = 0$. Choosing $Y_m = Y_{m,k}, k = 1, \cdots, d(m)$, elements of the basis in the space of all harmonics of degree m, one can write the condition b) in the equivalent form:

$b')$ $\int\limits_{\Gamma} h(\theta,t)Y_{m,k}(\theta)j_{\frac{n-2}{2}}(\lambda t)t^{n-1}dtdA(\theta)=0.$

This can be rephrased as follows:

$b'')$ $\widehat{h}_{m,k}(\lambda)=0$, for all zeros λ of the Bessel function $j_{\frac{n+m-2}{2}}(\lambda)$, where

$$\widehat{h}_{m,k}(\lambda)=\int\limits_{0}^{\infty} g_{m,k}(t)j_{\frac{n-2}{2}}(\lambda\ t)t^{n-1}dt$$

is Fourier-Bessel transform of $h_{m,k}$, which is a Fourier coefficient h:

$$h_{m,k}(t)=\int\limits_{S} h(y,t)Y_{m,k}(\theta)dA(\theta).$$

Theorem B.1 can be reformulated in terms of Darboux equation. Namely, since the spherical means $G=\mathcal{M}(f)$ is the unique solution for the Darboux equation (3.3), Theorem B.1 is equivalent to:

Theorem B.2 *Let h be a function defined on the cylinder Γ. Then the following statements are equivalent:*

i) There exists $f\in C_0^\infty(\overline{B})$ such that the following problem has a solution:

$$\begin{cases}\mathcal{D}G(x,t)=0,\ (x,t)\in\mathbb{R}^n\times\mathbb{R}_+,\\ G(x,t)=h(x,t),\ (x,t)\in\Gamma,\\ G(x,0)=f(x),\ G_t(x,0)=0,\ x\in\mathbb{R}^n.\end{cases}\tag{3.5}$$

ii) The conditions a) and b) of Theorem B.1 hold.

The proof of the implication $i)\Rightarrow ii)$ is quite simple and can be found in [4]. In the rest of this article, we will prove the converse implication.

2. Theorem B.2 as an extendibility property

Let $\mathcal{C} = B \times \mathbb{R}_+, \Gamma = S \times \mathbb{R}_+$. Consider the internal Darboux equation

$$\mathcal{D}H(x,t) = 0, \text{ for all } (x,t) \in \mathcal{C}, \tag{3.6}$$

with the the boundary condition

$$H(x,t) = h(x,t), \text{ for all } (x,t) \in \Gamma. \tag{3.7}$$

If $h \in C_0^\infty(S \times [0,2])$, the internal problem (3.6),(3.7) has a unique solution $H \in C^\infty(\overline{B} \times \mathbb{R}_+)$ such that $H(x,t) = 0$ for $t > 2$. Indeed, we notice that the equation (3.6) is nonsingular for all $t > 0$. Consider the time reversed problem for equations (3.6) and (3.7) with initial values $H(x,2) = H_t(x,2) = 0$. Since the boundary and initial conditions are compatible on $S \times \{2\}$, we obtain a unique solution $H \in C^\infty(\overline{B} \times (0,2])$. Extending H by zero for $t > 2$, we obtain the desired solution $H \in C^\infty(\overline{B} \times \mathbb{R}_+)$.

We now interpret *i)* in Theorem B.2 as an extendibility property of H. Indeed, suppose that a global solution G for (3.5) exists. Then $G(x,t) = \mathcal{M}(f)(x,t)$ and it solves (3.6) and (3.7). Since f is supported in $\overline{B}$, $G(x,t) = 0$ for all $x \in B$ and $t \geq 2$. Due to the uniqueness of H, we have $H(x,t) = G(x,t)$ for all $(x,t) \in \overline{B} \times \mathbb{R}_+$. That means, G is the global extension of H as the solution of the Darboux equation (3.5).

The above argument shows that the implication *ii)* $\Rightarrow$ *i)* in Theorem B.2 is equivalent to the following extendibility result:

Theorem B.3 *Assume that h satisfies condition ii) in Theorem B.2 (i.e, condition a) and b) in Theorem B.1). Let* $H \in C^\infty(\overline{B} \times \mathbb{R}_+)$ *be the unique solution for the internal problem (3.6) and (3.7) satisfying* $H(x,t) = 0$ *for* $t > 2$*. Then* H *extends to a global solution* G *of the Darboux equation (3.5), for some function* $f \in C_0^\infty(\overline{B})$.

C. Proof of Theorem B.3

We now present the first step in our proof, which shows that H extends to $t = 0$:

Proposition C.1 *Assume the condition in Theorem B.3. Then, there is a smooth function $f(x)$ in B such that $\lim_{t \to 0^+} H(x,t) = f(x)$, and $\lim_{t \to 0^+} H_t(x,t) = 0$, for all $x \in B$.*

This result, indeed, follows from a more general result in [4], which is stated shortly. Let B be a bounded domain [2] in $\mathbb{R}^n$ with smooth boundary S, $T > 0$, and $h \in C_0^\infty(S \times [0,T])$. Similar to the argument in the previous section, the boundary value internal problem (3.6), (3.7) has a unique solution $H \in C^\infty(\overline{B} \times \mathbb{R}_+)$ satisfying $H(x,t) = 0$ for all $t > T$. The following result is proven in [4, Lemma 14]:

Theorem C.2 *Let Δ_D be the Laplacian in B with Dirichlet boundary condition. Then, the following statements are equivalent:*

a) The boundary value h satisfies the orthogonality condition

$$\int_{\partial B} \int_0^\infty h(x,t) \partial_{\nu_x} \varphi_\lambda(x) j_{\frac{n-2}{2}}(\lambda t) t^{n-1} dA(x) dt = 0, \tag{3.8}$$

for all pairs of eigenvalue-eigenfunctions $(-\lambda^2, \varphi_\lambda)$ of Δ_D.

b) There is a smooth function $f(x)$ in B such that $\lim_{t \to 0^+} H(x,t) = f(x)$, and $\lim_{t \to 0^+} H_t(x,t) = 0$.

The detailed proof can be found in [4], and here we only briefly explain its main idea. The implication b) $\Rightarrow$ a) immediately follows from Stokes' formula applied in x variable and integration by parts with respect to t. Vice versa, assume that the orthogonality condition (3.8) holds. Since $H \in C^\infty(\overline{B} \times \mathbb{R}_+)$, one only needs to show

[2] Here, we do not need B to be the unit ball.

that H is not singular at $t = 0$ and $H_t(x, 0) = 0$. This is done by applying Stokes' formula in the domain $B \times (\epsilon, \infty], \epsilon \geq 0$, using the orthogonality condition (3.8) and letting $\epsilon \to 0$.

The next result shows that the function f obtained in Proposition C.1 vanishes up to infinite order on S:

Proposition C.3 *Let $H \in C^\infty(\overline{\mathcal{C}})$ be a solution of Darboux equation*

$$\mathcal{D}H(x,t) = 0, \ \forall (x,t) \in \mathcal{C}$$

such that $H(x,t) = 0$ for $t > 2$ and $x \in B$. If the boundary value $h = H|_\Gamma$ belongs to $C_0^\infty(S \times [0,2])$ then $f(x) = H(x,0)$ vanishes to infinite order on the sphere $|x| = 1$.

This is the main technical challenge of this chapter. We will present its proof in next section.

Proposition C.4 *Let $H \in C^\infty(\overline{\mathcal{C}})$ be a solution of Darboux equation*

$$\mathcal{D}H(x,t) = 0, \quad (x,t) \in \mathcal{C},$$

and $H(x,t) = 0$ for all $t > 2$ and $x \in B$. If $f(x) := H(x,0)$ vanishes to infinite order on the unit sphere $|x| = 1$, then H extends to a global solution $G(x,t)$ of Darboux equation. This global solution is given by the spherical means $G(x,t) = \mathcal{M}(f^*)(x,t)$ of the function f^* in $\mathbb{R}^n$ obtained from f by the zero extension outside of the unit ball.

Proposition C.4 was proven in [4]. We will reprove it in the next section for the sake of completeness. Meanwhile, we bind the above results together to form the proof of Theorem B.3:

The proof of Theorem B.3 Let $H(x,t)$ be the internal solution as in Theorem B.3. Due to Proposition C.1, H can be extended to a function $H \in C^\infty(\overline{\mathcal{C}})$. Then Proposition C.3 implies that $f(x) := H(x,0)$ vanishes to infinite order on the unit sphere $|x| = 1$. By Proposition C.4, H extends to a global solution $G(x,t) = \mathcal{M}(f^*)(x,t)$ for the Darboux equation (3.5) with smooth initial value f^*, which is obtained from f by zero extension outside B. This proves Theorem B.3.

D. Proofs of Propositions C.4 and C.3

1. Proposition C.4

As we have already mentioned, Proposition C.4 is proven in [4]. We will present the proof here for the sake of completeness.

Since $f(x) = H(x,0)$ vanishes to infinite order on the boundary of B, its zero extension f^* belongs to $C_0^\infty(\mathbb{R}^n)$. Then the natural candidate for the extended solution is given by the spherical means of f^*: $G = \mathcal{M}(f^*)$. This function is globally defined and, since f^* is smooth, belongs to $C^\infty(\mathbb{R}^n \times \mathbb{R}_+)$. It is a global solution of Darboux equation and our goal is to prove that H and G coincide in the solid cylinder $\overline{\mathcal{C}} := \overline{B} \times [0,\infty)$.

First observe that both solutions share the same initial data at $t = 2$:

$$G(x,2) = H(x,2) = G_t(x,2) = H_t(x,2) = 0.$$

Then, due to domain of dependence theorem [11, p.696],

$$G(x,t) = H(x,t) = 0 \text{ for all } (x,t) \in \mathcal{K}^+, \tag{3.9}$$

where $\mathcal{K}^+$ is the upward characteristic cone (see Fig. 2)

$$\mathcal{K}^+ := \{(x,t) \in \overline{B} \times [0,2] : t - |x| \geq 1\}.$$

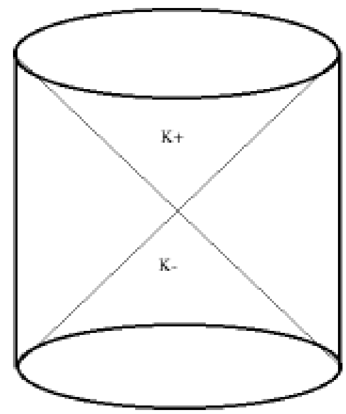

Fig. 2. Picture of the double cone $\mathcal{K}$ inside the cylinder $\mathcal{C}$: the upper one is $\mathcal{K}_+$, the lower one is $\mathcal{K}_-$.

Moreover, $G(x,t)$ and $H(x,t)$ also share the initial data at $t=0$ and $x \in B$:

$$G(x,0) = H(x,0) = f(x), G_t(x,0) = H_t(x,0) = 0.$$

Therefore, again by the dependence domain theorem, they coincide in the downward characteristic cone with the base $\overline{B} \times \{0\}$ (see Fig. 2):

$$\mathcal{K}^- := \{(x,t) \in \overline{B} \times [0,1] : |x| + t \leq 1\}.$$

Hence, the difference $U(x,t) := G(x,t) - H(x,t)$ vanishes in the union $\mathcal{K} = \mathcal{K}^+ \cup \mathcal{K}^-$ of the two cones. Besides, since both G and H vanish for $t > 2$, so does U.

Since U satisfies Darboux equation inside the cylinder $\mathcal{C}$, its Fourier-Bessel transform

$$\widehat{U}(x,\alpha) = \int_0^\infty U(x,t) j_{\frac{n-2}{2}}(\alpha t) t^{n-1} dt$$

satisfies the Helmholtz equation:

$$\Delta_x \widehat{U}(x,\alpha) = -\alpha^2 \widehat{U}(x,\alpha),\ \forall x \in B.$$

Hence, $\widehat{U}(x,\alpha)$ is real analytic with respect to $x \in B$.

We recall that $U(x,t) = 0$ in $\mathcal{K} \cup (B \times [2,\infty))$. The union $\mathcal{K} \cup (B \times [2,\infty))$ contains the entire ray $\{(0,t) : 0 \leq t < \infty\}$ and hence after taking Fourier-Bessel transform one concludes that $\widehat{U}(0,\alpha) = 0$. Since U is smooth, the same argument can be applied to $D_x^\beta U$ to obtain:

$$D_x^\beta \widehat{U}(0,\alpha) = \widehat{D_x^\beta U}(0,\alpha) = 0.$$

Thus, $\widehat{U}(.,\lambda)$ vanishes to infinite order at $x = 0$. Since $\widehat{U}(.,\lambda)$ is real-analytic, one concludes that $\widehat{U}(x,\alpha) = 0$, for all $x \in B$ and $\alpha \geq 0$. Taking inverse Fourier-Bessel transform, we obtain $U = 0$ and therefore $G = H$ in $\mathcal{C}$. This completes the proof of Proposition C.4.

2. Proof of Proposition C.3

We want to prove that $f := H(.,0)$ vanishes on the sphere S to infinite order:

$$D_x^\beta f(x) = 0, \quad \forall x : |x| = 1.$$

First of all, recall that $G(x,2) = G_t(x,2) = 0$ implies $H = 0$ in the upward characteristic cone $\mathcal{K}^+$:

$$H(x,t) = 0,\ |x| \leq t-1, 1 \leq t \leq 2.$$

In particular, H vanishes at the vertex of the cone $\mathcal{K}^+$:

$$H(0,1) = 0.$$

Since H is smooth in the solid cylinder $\overline{\mathcal{C}} := \overline{B} \times [0, \infty)$, the same conclusion holds for for all derivatives of H:

$$D_t^j D_x^\beta H(0,1) = 0. \tag{3.10}$$

Here $j = 0, 1, ...$ and β is an arbitrary multi-index.

Now we relate H to the spherical means of the initial value $f(x)$. Albeit we cannot assert so far that $H = \mathcal{M}(f)$, we can claim that the two functions coincide at least in the downward characteristic cone $\mathcal{K}^-$:

$$\mathcal{K}^- = \{|x| < 1 - t,\ 0 \leq t \leq 1\}.$$

Indeed, both H and $\mathcal{M}(f)$ solve the equation $\mathcal{D}G = 0$ on $\mathcal{C}$ and share the same initial values on $\overline{B}$: $G(x, 0) = f, G_t(x, 0) = 0$. The conclusion now follows from the domain of dependence argument.

Since $(0, 1)$ is the vertex of $\mathcal{K}^-$, due to (3.10), we conclude that

$$(D_t^j D_x^\beta \mathcal{M} f)(0,1) = 0. \tag{3.11}$$

Now our aim is to derive from (3.11) that $f(x)$ vanishes on the sphere $|x| = 1$ along with all derivatives.

To this end, first observe that condition (3.11) is invariant with respect to action of the orthogonal group $O(n)$ and hence it holds for any term $f_{m,k}(r) r^m Y_{m,k}(\theta)$ in Fourier series of f:

$$f(x) = f(r\theta) = \sum_{m=0}^{\infty} \sum_{k=1}^{d(m)} f_{m,k}(r) r^m Y_{m,k}(\theta), \tag{3.12}$$

where $r = |x|, |\theta| = 1$ and $\{Y_{m,k}, k = 1, ..., d(m)\}$ is the orthonormal basis in the space of all spherical harmonics of degree m. Due to smoothness of f in the closed ball $\overline{B}$, the series (3.12) converges uniformly with all derivatives and hence it suffices to prove that each term vanishes on the unit sphere to infinite order.

Thus, we can assume that f is just a single term:

$$f(x) = f_m(r)P_m(x), \tag{3.13}$$

where

$$P_m(x) = r^m Y_{m,k}(\theta)$$

is a spatial harmonic of degree m. To prove that f vanishes to infinite order for $|x| = 1$, it suffices to prove that all the derivatives $f_m^{(j)}$ vanishes at $t = 1$:

$$f_m^{(j)}(1) = 0, j = 0, 1, ...$$

We will prove this by constructing a system of linear equations that these numbers satisfy.

Lemma D.1 *The following identities hold*

1. *For any* $i \geq 0$,

$$(\frac{d^i}{dr^i}\mathcal{L}_m f_m)(1) = 0, \tag{3.14}$$

where $\mathcal{L}_m$ *is the following differential operator of order* m*:*

$$\mathcal{L}_m = \prod_{s=1}^{m} \left(\frac{1}{n + 2(m-s)} r \frac{d}{dr} + 1 \right). \tag{3.15}$$

2. *For any* $l \geq 0$,

$$(\mathcal{Q}_m)^l (f_m)(1) = 0, \tag{3.16}$$

where $\mathcal{Q}_m$ *is the following differential operator of order* 2*:*

$$\mathcal{Q}_m = \partial_r^2 + \frac{n + 2m - 1}{r} \partial_r. \tag{3.17}$$

Proof The identity (3.14) is closely related to [14, Theorem 2.1]. We follow here the proof of the later to show (3.14). Let us introduce the operator

$$\pi_m(g)(r) = \sum_{k=1}^{d(m)} g_{m,k}(r) r^m Y_k^m(\theta),$$

the projection onto spherical harmonics of degree m. We observe that differentiation in x and the transform $\mathcal{M}$ commute:

$$D_x^\beta \mathcal{M}(f)(x,t) = \mathcal{M}(D^\beta f)(x,t).$$

Since the spherical mean with the center at $x = 0$ and radius $t = 1$ is exactly the projection onto order zero harmonics (constants), identity (3.11) now reads as

$$\frac{d^i}{dr^i}\pi_0(D^\beta f)(1) = 0 \tag{3.18}$$

for all $i \in Z_+$ and all multiindices β.

The projection π_0 of the derivatives of f of the form (3.13) was computed in [14, formula (2.10)]:

$$\pi_0(D^\beta f) = \left(D^\beta P_m\right)(\mathcal{L}_m f_m),\ m = |\beta| = \beta_1 + \cdots + \beta_n,$$

where the differential operator $\mathcal{L}_m$ is defined in (3.15). Since P_m is a polynomial of degree m, we can choose in this formula the multiindex β so that $D^\beta P_m$ is a non-zero constant. By allowing the index i to be arbitrary, we derive from (3.18) the identity (3.14).

As for the identity (3.16), it comes from the equation $\mathcal{D}H = 0$, which means

$$\mathcal{B}H(x,t) = \Delta_x H(x,t),\ \forall (x,t) \in \mathcal{C},$$

where $\mathcal{B}$ is the Bessel operator acting on $t-$variable:

$$\mathcal{B} = \partial_t^2 + \frac{n-1}{t}.$$

Iterating the above identity, one obtains

$$\mathcal{B}^l H(x,t) = \Delta^l H(x,t),\ \forall (x,t) \in \mathcal{C}.$$

Since $H \in C^\infty(\overline{\mathcal{C}})$, the above equality holds up to the boundary Γ. In particular, since $G(x,t) = h(x,t)$ for all $(x,t) \in \Gamma$ and $G(x,0) = f(x)$, we have

$$\mathcal{B}^l h(x,0) = \Delta^l f(x),\ \forall x \in S. \tag{3.19}$$

Since h vanishes to infinite order at $t = 0$, one concludes that for all $x \in S$

$$\Delta^l f(x) = 0, l = 0, 1, ...$$

Now, taking into account that the Laplacian Δ acts on m^{th}-harmonic term as the operator

$$\mathcal{Q}_m := \partial_r^2 + \frac{n-1+2m}{r}\partial_r,$$

we arrive at

$$\mathcal{Q}_m^l f_m = 0, l = 0, 1, ... \tag{3.20}$$

∎

Let us apply Lemma D.1 when indices j and l run independently from 0 to $m-1$. We can write the differential operators in (3.14) and (3.16) in the form

$$\frac{d^i}{dr^i}\mathcal{L}_m = A_{i,0}(r) + A_{i,1}(r)\frac{d}{dr} + \cdots + A_{i,i+m}(r)\frac{d^{i+m}}{dr^{i+m}},$$

and [3]

$$\mathcal{Q}_m^l = B_{m-1-l,0}(r) + B_{m-1-l,1}(r)\frac{d}{dr} + \cdots + B_{m-1-l,2m-1}(r)\frac{d^{2m-1}}{dr^{2m-1}}.$$

Consider the vector

$$F := (f_m(1), f_m'(1), \cdots, f_m^{(2m-1)}(1)),$$

which consist of the first $2m$ derivatives (including that of order 0) of f_m at the point $r = 1$.

Let $i = 0, 1, \cdots, m-1$ and $l = 0, 1, \cdots, m-1$ in Lemma D.1, we conclude that F satisfies the following $2m \times 2m$ linear system:

$$\begin{cases} A_{0,0}F_0 + A_{0,1}F_1 + \cdots + A_{0,m}F_m = 0 \\ A_{1,1}F_1 + A_{1,2}F_2 + \cdots\cdots\cdots + A_{1,m+1}F_{m+1} = 0 \\ \cdots\cdots\cdots \\ A_{m-1,m-1}F_{m-1} + A_{m-1,m}F_m + \cdots\cdots\cdots + A_{m-1,2m-1}F_{2m-1} = 0 \\ B_{0,0}F_0 + B_{0,1}F_1 + \cdots\cdots\cdots\cdots\cdots\cdots\cdots + B_{0,2m-1}F_{2m-1} = 0 \\ B_{1,0}F_0 + B_{1,1}F_1 + \cdots\cdots\cdots\cdots\cdots\cdots\cdots + B_{1,2m-1}F_{2m-1} = 0 \\ \cdots\cdots\cdots \\ B_{m-1,0}F_0 + B_{m-1,1}F_1 + \cdots\cdots\cdots\cdots\cdots\cdots + B_{m-1,2m-1}F_{2m-1} = 0 \end{cases} \tag{3.21}$$

Here the matrix coefficients are $A_{i,j} = A_{i,j}(1), B_{i,j} = B_{i,j}(1)$.

Lemma D.2 *The linear $2m \times 2m-$ system (3.21) is nondegenerate.*

We will prove this lemma later. Assuming that the lemma is proven, we can complete the proof of Proposition C.3. Since the system (3.21) is nondegenerate, one concludes

[3]Many elements $B_{i,j}$ are zero, but we consider the operator $\mathcal{Q}_m^l$ in the above form for the sake of simplicity.

that the first $2m$ derivatives of f_m vanish:

$$f_m^{(j)}(1) = F_j = 0, 0 \leq j \leq 2m-1.$$

To obtain the vanishing of higher order derivatives, we will exploit higher values for the index i in (3.14). Choosing $i = m$ in (3.14) results in shifting of the vector of the unknowns to the right:

$$A_{m,0}F_m + \cdots + A_{m,2m-1}F_{2m-1} + A_{m,m}F_{2m} = 0$$

which along with $F_0 = \cdots = F_{2m-1} = 0$ implies $F_{2m} = 0$ because $A_{m,m} \neq 0$. Then the next choice $i = m+1$ leads to $F_{2m+1} = 0$. Proceeding this way by taking successively $j = m, m+1, m+2, \cdots$ one obtains $F_\nu = 0$ for all $\nu \geq 0$.

This completes the proof of Proposition C.3. ∎

Remark D.3 *As it was mentioned earlier, Proposition C.3 was proved for odd n in [4]. The proof used Weyl and Poisson-Sonine integral transforms, applied to the solution $H(x,t)$ in t and x variables correspondingly. To have control over the derivatives of $H(x,0)$ on the sphere $|x| = 1$ one needs the inverse transforms to be local (differential) operators which is the case only in odd dimensions. That is why the proof in [4] did not work in even dimensions.*

Proof of Lemma D.2 Let us denote by A_i the *ith* row of the matrix of the first m equations in the system (3.21) and by B_i the *ith* row from the second group of m linear equations in (3.21).

One observes that the vectors $A_i, i = 0, \cdots, m-1$ are linearly independent, as $A_{i,j} = 0$ for all $j > m+i$ and $A_{i,m+i} \neq 0$.

We now use induction with respect to the length of the system of the vectors. Namely, we will show that adding successively each vector $B_0, \cdots, B_{m-1}$ to the set

$\{A^0, .., A^{m-1}\}$ does not break the linear independence. Then, in m steps, we will obtain the linear independence of the entire system.

Thus, our inductive assumption is that the system

$$\mathcal{S}_p := \{A_1, \cdots, A_{m-1}, B_0, \cdots, B_{p-1}\}$$

is linearly independent for some $p \leq m-1$. Now we want to check that it remains linearly independent after adding the next vector B_p. In other words, the vector B_p is linearly independent from the set $\mathcal{S}_p$.

To this end, it suffices to find a vector $v_p \in \mathrm{I\!R}^{2m}$ that is orthogonal to $\mathcal{S}_p$ but not to B_p:

$$\begin{aligned} &\langle A_i, v_p \rangle = 0, i = 0, \cdots, m-1, \\ &\langle B_j, v_p \rangle = 0, j = 0, \cdots, p-1, \\ &\langle B_p, v_p \rangle \neq 0. \end{aligned} \tag{3.22}$$

Indeed, we take the function $\Psi_p(r) = r^{-n-2p}$ and construct the vector of successive derivatives at $r = 1$:

$$v_p = (\Psi_p(1), \cdots, \Psi_p^{(2m-1)}(1)).$$

Recall that

$$\mathcal{L}_m = \prod_{s=1}^{m} \left(\frac{1}{n+2(m-s)} r \frac{d}{dr} + 1 \right).$$

Since all the first order differential operators in the above product commute, we can rewrite

$$\mathcal{L}_m = \left[\prod_{s \neq m-p}^{m} \left(\frac{1}{n+2(m-s)} r \frac{d}{dr} + 1 \right) \right] \left(\frac{1}{n+2p} r \frac{d}{dr} + 1 \right).$$

A simple observation gives

$$\left(\frac{1}{n+2p} r \frac{d}{dr} + 1 \right) \Psi_p(r) = 0, \forall r > 0.$$

Therefore,

$$\mathcal{L}_m \Psi_p(r) = 0, \ \forall r > 0.$$

This, in particular, implies

$$\left(\frac{d^i}{dr^i}\mathcal{L}_m \Psi_p\right)(1) = 0, \ i = 0, 1, \cdots, m-1.$$

By the definition of A_i, this is equivalent to the first group of equations in (3.22):

$$\langle A_i, v_p \rangle = 0, i = 0, \cdots, m-1.$$

Here, as we defined, $v_p = (\Psi_p(1), \cdots, \Psi_p^{(2m-1)}(1))$ is the vector of successive derivatives of Ψ_p evaluated at $r = 1$.

We recall that the construction of B_i comes from the iteration of the differential operator

$$\mathcal{Q}_m = \partial_r^2 + \frac{n-1+2m}{r}\partial_r$$

evaluated at $r = 1$. Straightforward computation yields

$$\mathcal{Q}_m^l \Psi_p(r) = \begin{cases} C_l r^{-n-2(p+l)}, \ 0 \leq l \leq m-1-p, \\ 0, \ m-p \leq l \leq m-1. \end{cases}$$

Here C_l are nonzero constants.

Substituting $r = 1$, due to the definition of B_i, one is led to the remained equations in (3.22):

$$\langle B_i, v_p \rangle = \mathcal{Q}_m^{(m-1-i)} \Psi_p(1) = 0, \forall i = 0, .., p-1,$$

and

$$\langle B_p, v_p \rangle = \mathcal{Q}_m^{(m-1-p)} \Psi_p(1) = C_{m-1-p} \neq 0.$$

This completes the proof of Lemma D.2 and thus finishes the proof of Theorem B.1.

CHAPTER IV

STABILITY ANALYSIS

In this chapter we present some stability analysis for the reconstruction in TAT. The analysis relies on the propagation of singularities of the solution $u(x,t)$ for the wave equation. The visibility condition requires all the singularities of f propagate to the observation surface S. It has been previously shown by various authors that under this condition, the reconstruction is Lipschitz stable. We show that if the condition is violated, the reconstruction of TAT is not even Hölder stable.

A. Introduction

Let us recall the commonly accepted model of TAT:

$$\begin{cases} u_{tt}(x,t) - c^2(x) \triangle u(x,t) = 0, \quad x \in \mathbb{R}^n, \quad t \geq 0, \\ u(x,0) = f(x), \quad u_t(x,0) = 0, \\ u(y,t) = g(y,t), y \in S, t \geq 0. \end{cases} \tag{4.1}$$

Here $c(x)$ is the the ultrasound speed at location x, $g(y,t)$ is the pressure measured at location $y \in S$ and time t. We assume that the sound speed $c(x)$ is smooth and there exists constants $C_0, c_0 > 0$ such that $c_0 \leq c(x) \leq C_0$ for all $x \in \mathbb{R}^n$.

In this chapter, we concentrate on the stability analysis of the reconstruction of f in a region of interest Ω, which is a bounded open domain in $\mathbb{R}^n$, from the measured data g on an observation surface [1] S. This issue has been partially addressed in a number of papers [67, 68, 39, 20, 31, 55, 30, 48, 42].

If the speed c is constant, it was shown that under the so-called *visibility* condition

[1] Here S need not be the boundary of the region of interest Ω. We also do not assume any special geometries for S and Ω.

the reconstruction is only mildly unstable, similarly to the inversion of the standard Radon transform (see, e.g. [39], for more discussion and references). The visibility condition requires that for each $x \in \Omega$, every straight line passing through x intersects the observation surface S. Equivalently, it requires Ω to lie inside the convex hull of S in $\mathbb{R}^n$. For instance, if S is a closed hypersurface surrounding Ω then the visibility condition is satisfied.

Let us now consider the case of variable speed $c(x)$. If Ω is enclosed by S, it was argued and demonstrated by numerics in [31] that if some geometric rays passing through Ω are trapped inside S, then the singularities of f that lead to these rays cannot be stably reconstructed. On the other hand, the authors of [55] recently proved that under visibility condition, which can be defined for variable speed using geometric rays instead of straight lines, the reconstruction is Lipschitz stable. Our goal is to prove a complementary result, which shows that the reconstruction is not Hölder stable if the visibility condition does not hold. The visibility condition can be violated when either the data is incomplete (S does not completely surround Ω, see, e.g., [39]), or the trapping phenomenon occurs (e.g., [31]). These two cases have the same instability, and we do not distinguish between them. Although we do not use this in the text, it should be mentioned that the Lipschitz instability can be obtained from the general framework proposed in [54].

The chapter is organized as follows. In section B, we recall the notion of wavefront set, whose propagation is the central issue of stability analysis in TAT (as well as other types of tomography, e.g., [42, 50, 19, 26, 27]). We then present our instability result in section C.

B. Singularities and wavefront set

In tomography, singularities are usually related to sharp details, for instance the boundaries of objects, jumps in densities, or interfaces between tissues. In many cases, singularities (rather than the exact image) are of the main interest (e.g., they are the objects of reconstruction in local tomography, e.g. [37, 17, 16]). Reconstruction of singularities in (X-ray) tomography was first investigated by Greenleaf and Uhlmann [26] and Quinto [50]. We will see that propagation of singularities also plays an important role in the stability analysis of reconstruction in TAT.

We denote by $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$ the standard spaces of test functions and distributions on $\mathbb{R}^n$. We now recall the definition of wavefront set (e.g., [57]):

Definition B.1 *Let $f \in \mathcal{D}'(\mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$, and $\xi_0 \in \mathbb{R}^n \setminus \{0\}$. Then f is microlocally smooth at (x_0, ξ_0) if there is a function $\varphi \in C_0^\infty(\mathbb{R}^n)$ satisfying $\varphi(x_0) \neq 0$ and an open cone V containing ξ_0, such that $\mathcal{F}(\varphi f)$ is rapidly decreasing in V. That is, for any $N > 0$ there exists a constant C_N such that*

$$|\mathcal{F}(\varphi f)(\xi)| \leq C_N \langle \xi \rangle^{-N}, \text{ for all } \xi \in V.$$

Here $\mathcal{F}$ is the Fourier transform and $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. The **wavefront set** *of f, denoted by $WF(f)$, is the complement of the set of all $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ where f is microlocally smooth.*

For example, if f is the characteristic function of an open set Ω with smooth boundary $\partial\Omega$, then $(x_0, \xi_0) \in WF(f)$ if and only if $x_0 \in \partial\Omega$ and ξ_0 is perpendicular to the tangent plane $T_{x_0}\partial\Omega$ of $\partial\Omega$ at x_0.

Propagation of the wavefront set of the solution $u(x,t)$ for equation (4.1) can be precisely described (e.g., [45, 57, 55, 56]). Since the initial velocity $u_t(x, 0)$ is zero, each element $(x_0, \xi_0) \in WF(f)$ propagates in two opposite directions ξ_0 and $-\xi_0$. Let us consider the bi-characteristics $(x_\pm(s), t_\pm(s), \xi_\pm(s), \tau_\pm(s))$, which are the solutions of the following Hamiltonian systems:

$$\begin{cases} \dot{x}(s) = -c^2(x)\xi, \ \dot{t}(s) = \tau, \\ \dot{\xi}(s) = \frac{1}{2}\nabla c^2(x)|\xi|^2, \ \dot{\tau}(s) = 0 \\ (x(0), t(0), \xi(0), \tau(0)) = (x_0, 0, \mp\xi_0, c(x_0)|\xi_0|). \end{cases}$$

Since $c(x)$ is smooth and $0 < c_0 \leq c(x) \leq C_0$, these bi-characteristics are well defined on $s \in \overline{\mathbb{R}}_+$, and $(x_\pm(s), t_\pm(s), \xi_\pm(s), \tau_\pm(s)) \in WF(u)$ for all s. We denote the (x,t)-projections of these bicharacteristics by $\mathcal{R}_+(x_0, \xi_0)$ and $\mathcal{R}_-(x_0, \xi_0)$. The following result can be easily obtained by basic tools of microlocal analysis (e.g., [57]):

Theorem B.2 *Consider equation (4.1). Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n \times \overline{\mathbb{R}}_+$. Assume that for all $(x, \xi) \in WF(f)$, $\mathcal{R}_+(x, \xi)$ and $\mathcal{R}_-(x, \xi)$ do not intersect $\mathcal{O}$. Then $u \in C^\infty(\mathcal{O})$.*

Let $\mathcal{R}_x(x, \xi)$ be the x-projection of $\mathcal{R}_+(x, \xi) \cup \mathcal{R}_-(x, \xi)$. $\mathcal{R}_x(x, \xi)$ is, indeed, a connected smooth curve in $\mathbb{R}^n$, which we call a (geometric) ray. The following result is a simple consequence of Theorem B.2:

Corrolary B.3 *Consider equation (4.1). Let V be an open subset of $\mathbb{R}^n$. Assume that for all $(x, \xi) \in WF(f)$, the rays $\mathcal{R}_x(x, \xi)$ do not intersect V. Then $u \in C^\infty(V \times \overline{\mathbb{R}}_+)$.*

C. Instability of reconstruction in thermoacoustic tomography

Let us return to the main equation of TAT:

$$\begin{cases} u_{tt}(x,t) - c^2(x) \triangle u(x,t) = 0, \quad \forall x \in \mathbb{R}^n, \quad t > 0, \\ u(x,0) = f(x), \quad u_t(x,0) = 0. \end{cases} \tag{4.2}$$

Let S be a closed piece of a hypersurface in $\mathbb{R}^n$, $T > 0$, $\Gamma = S \times [0, T]$, and g be the restriction of u to Γ. We define the linear operator

$$\begin{aligned} \mathcal{T} : L^2(\Omega) &\longrightarrow \mathcal{D}'(\Gamma) \\ f &\longmapsto g. \end{aligned}$$

Here, we identify $L^2(\Omega)$ with the subspace of $L^2(\mathbb{R}^n)$ containing functions supported in $\overline{\Omega}$. Assuming $\mathcal{T}$ is injective, we now consider the stability problem of the reconstruction of f from g. Let U and V be open sets in $\mathbb{R}^n$ such that $U \subset \Omega$ and $S \subset V$.

Theorem C.1 *Assume that there exists a nonzero vector $\xi_0 \in \mathbb{R}^n \setminus 0$ such that for all $x \in U$, the rays $\mathcal{R}_x(x, \xi_0)$ do not intersect V. Then the reconstruction of f from g is not Hölder stable. That is, there do not exist $\mu > 0$, $\delta > 0$, $s_0, s_1 \geq 0$, and $C > 0$ such that*

$$\|f\|_{L^2(\Omega)} \leq C \|\mathcal{T} f\|^{\mu}_{H^{s_0}(\Gamma)}, \text{ for all } f \in H^{s_1}(\Omega) \text{ satisfying } \|f\|_{H^{s_1}(\Omega)} \leq \delta.$$

In order to prove this theorem, we need an auxiliary result. Without loss of generality, we can assume that $0 \in U$ and $\xi_0 = (0, .., 0, 1)$. Since U is open, there is an open set $U_0 \subset \mathbb{R}^{n-1}$ and $\varepsilon > 0$ such that $\overline{U}_0 \times \bar{I} \subset U$, where $I = (-\varepsilon, \varepsilon)$. Let us fix a nonzero function $f_0 \in C_0^\infty(U_0)$. For each $x \in \mathbb{R}^n$, we write $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. We now consider

$$X = \{f \in L^2(U) : f(x) = f_0(x')h(x_n), h \in L^2(\mathbb{R}), supp(h) \subset \overline{I}\}.$$

Then, X is an infinite dimensional closed subspace of $L^2(\Omega)$.

Lemma C.2 *For all $s \geq 0$, $\mathcal{T}$ induces a linear bounded operator*

$$\mathcal{T}|_X : X \longrightarrow H^s(\Gamma).$$

In what follows, we first prove Lemma C.2, and then show how to obtain Theorem C.1 from it.

1. Proof of Lemma C.2

Let $f \in X$, then $WF(f) \subset U \times \{\lambda\xi_0 : \lambda \neq 0\}$. Since $\mathcal{R}_x(x, \lambda\xi_0) = \mathcal{R}_x(x, \xi_0)$, one has $\mathcal{R}_x(x, \xi) = \mathcal{R}_x(x, \xi_0)$ for all $(x, \xi) \in WF(f)$. Assuming the condition in Theorem C.1, one deduces that $\mathcal{R}_x(x, \xi)$ do not intersect V for all $(x, \xi) \in WF(f)$.

We consider equation (4.2) and let $\mathcal{P}(f) = u|_{V \times \overline{\mathbb{R}}_+}$. Due to Corollary B.3, $\mathcal{P}|_X$ is a linear operator from X to $C^\infty(V \times \overline{\mathbb{R}}_+)$. Let V_0 be an open set in $\mathbb{R}^n$ such that $S \subset V_0 \subset \overline{V}_0 \subset V$. Then, for all $s \geq 0$ and $T > 0$, $\mathcal{P}$ induces a linear operator:

$$\mathcal{P}|_X : X \to H^s(V_0 \times [0, T]).$$

We now prove that this operator is bounded. Indeed, we first show that $\mathcal{P} : L^2(\Omega) \to L^2(V_0 \times [0, T])$ is bounded. Let $p(., t) = \int_0^t u(., s) ds$, then, from (4.2), p solves the equation:

$$\begin{cases} p_{tt} - c^2(x)\Delta p(x, t) = 0, \; x \in \mathbb{R}^n, \; t > 0, \\ p(x, 0) = 0, \; p_t(x, 0) = f(x). \end{cases} \tag{4.3}$$

Denoting $E(t) = \|c^{-1} p_t(., t)\|^2_{L^2(\mathbb{R}^n)} + \|\nabla p(., t)\|^2_{L^2(\Omega)}$, one concludes that E is independent of t, due to the conservation of energy (e.g., [11]). That is, $E(t) = E(0)$ for all $t \in \mathbb{R}_+$. Since $p_t(., t) = u(., t)$, $p(., 0) = 0$, and $p_t(., 0) = f$, one derives

$$\|c^{-1} u(., t)\|^2_{L^2(\mathbb{R}^n)} + \|\nabla p(., t)\|^2_{\mathbb{R}^n} = \|c^{-1} f\|_{L^2(\mathbb{R}^n)}.$$

Using the inequalities $0 < c_0 \leq c(x) \leq C_0$, one deduces that there is a constant $A > 0$ satisfying

$$\|u(., t)\|_{L^2(\mathbb{R}^n)} \leq A \|f\|_{L^2(\mathbb{R}^n)}, \quad \text{for all } f \in L^2(\Omega).$$

Hence,

$$\|u\|_{L^2(V_0\times[0,T])} \leq \|u(.,t)\|_{L^2(\mathbb{R}^n\times[0,T])} \leq AT\|f\|_{L^2(\mathbb{R}^n)},$$

which proves the boundedness of $\mathcal{P} : L^2(\Omega) \to L^2(V_0 \times [0,T])$.

Since $H^s(V_0 \times [0,T])$ is continuously embedded into $L^2(V_0 \times [0,T])$, applying Propositions D.1 (see appendix), one concludes that $\mathcal{P}|_X : X \to H^s(V_0 \times [0,T])$ is bounded.

We are now ready to prove Lemma C.2. We recall that the restriction $\mathcal{R}(u) = u|_\Gamma$, as a linear operator $\mathcal{R} : H^s(V_0 \times [0,T]) \longrightarrow H^{s-1/2}(\Gamma)$, is bounded for any $s > \frac{1}{2}$. Therefore,

$$\mathcal{T}|_X = \mathcal{R} \circ \mathcal{P}|_X : X \longrightarrow H^s(\Gamma)$$

is bounded for any $s \geq 0$.

2. Proof of Theorem C.1

We first recall some facts concerning singular value decomposition (e.g., [13, 25]). Let H_1, H_2 be Hilbert spaces and $A : H_1 \to H_2$ be a bounded injective operator. Let A^* be the adjoint operator of A and $B = A^*A$. Then $B : H_1 \to H_2$ is a positive definite bounded operator. Let us denote by $\{s_j^2\}$ the eigenvalues of B and by $\{e_j\}$ the corresponding unit norm eigenvectors. Then $\{e_j\}_j$ is an orthogonal basis of H_1. Denoting $f_j = \frac{1}{s_j}A(e_j)$, it is simple to show $\{f_j\}_j$ is an orthonormal set in H_2. If $range(A)$ is dense in H_2, then $\{f_j\}_j$ is an orthonormal basis of H_2.

For an operator A, we denote by $\{s_j(A)\}_j$ the set of above s_j, which are chosen to be positive and decreasing. Then $\{s_j(A)\}_j$ are called **s-numbers** (or **singular values**) of A. We will need the following asymptotic behavior of s-numbers:

Lemma C.3 *(e.g., [13, page 119])*

1. *Let $s_1 > s_2$ and $i_1 : H^{s_1}(\Gamma) \longrightarrow H^{s_2}(\Gamma)$ be the natural embedding. There exists*

a constant $c_1 > 0$ independent of j, such that $s_j(i_1) \leq c_1 j^{\frac{s_2 - s_1}{n}}$.

2. *Let $s > 0$, $\varepsilon > 0$, and $i_2 : H_0^s(-\varepsilon, \varepsilon) \longrightarrow L^2(-\varepsilon, \varepsilon)$ be the natural embedding. There exists a constant $c_2 > 0$ independent of j, such that $s_j(i_2) \geq c_2 j^{-s}$.*

Proof of Theorem C.1 Suppose that there exist $\mu > 0$, $\delta > 0$, $s_0, s_1 \geq 0$, and $C > 0$ such that

$$\|f\|_{L^2(\Omega)} \leq C \|\mathcal{T} f\|^{\mu}_{H^{s_0}(\Gamma)}$$

for all $f \in H^{s_1}(\Omega)$ satisfying $\|f\|_{H^{s_1}(\Omega)} \leq \delta$. Then for any $f \in H^{s_1}(\Omega)$, applying this inequality to $\frac{\delta f}{\|f\|_{H^{s_1}(\Omega)}}$, we have

$$\frac{\|f\|_{L^2(\Omega)}}{\|f\|_{H^{s_1}(\Omega)}} \leq C_1 \left(\frac{\|\mathcal{T} f\|_{H^{s_0}(\Gamma)}}{\|f\|_{H^{s_1}(\Omega)}} \right)^{\mu},$$

where C_1 is a constant independent of f. That is,

$$\|f\|_{L^2(\Omega)} \leq C_1 \|\mathcal{T} f\|^{\mu}_{H^{s_0}(\Gamma)} \|f\|^{1-\mu}_{H^{s_1}(\Omega)}, \tag{4.4}$$

for all $f \in H^{s_1}(\Omega)$. Let us consider the following subspace of space X from Lemma C.2:

$$X_{s_1} = \{f : f(x) = f(x', x_n) = u_0(x') h(x_n), h \in H_0^{s_1}(I)\}.$$

We then have $X_{s_1} \subset X \cap H^{s_1}(\Omega)$. We now prove that (4.4) cannot be true for all $f \in X_{s_1}$. Indeed, due to Lemma C.2, for any $f \in X_{s_1} \subset X$, one has

$$\|\mathcal{T} f\|^{\mu}_{H^s(\Gamma)} \leq C_s \|f\|^{\mu}_{L^2(\Omega)}. \tag{4.5}$$

Here, $s \geq 0$ and C_s is a general constant depending on s. Combing (4.4) and (4.5), we have

$$\|Tf\|^{\mu}_{H^s(\Gamma)} \|f\|^{1-\mu}_{L^2(\Omega)} \leq C_s \|f\|_{L^2(\Omega)} \leq C_s \|\mathcal{T} f\|^{\mu}_{H^{s_0}(\Gamma)} \|f\|^{1-\mu}_{H^{s_1}(\Omega)}.$$

That is,

$$\left(\frac{\|f\|_{L^2(\Omega)}}{\|f\|_{H^{s_1}(\Omega)}}\right)^{1-\mu} \le C_s \left(\frac{\|\mathcal{T}f\|_{H^{s_0}(\Gamma)}}{\|\mathcal{T}f\|_{H^{s}(\Gamma)}}\right)^{\mu}. \tag{4.6}$$

Since $f(x) = f_0(x')h(x_n)$, direct computations show

$$\begin{aligned} \|f\|_{L^2(\Omega)} &= \|f_0\|_{L^2(\mathbb{R}^{n-1})}\|h\|_{L^2(I)}, \\ \|f\|_{H^{s_1}(\Omega)} &\le \|f_0\|_{H^{s_1}(\mathbb{R}^{n-1})}\|h\|_{H^{s_1}(I)}. \end{aligned}$$

Inequality (4.6) gives

$$\left(\frac{\|h\|_{L^2(I)}}{\|h\|_{H^{s_1}(I)}}\right)^{1-\mu} \le C_s \left(\frac{\|\mathcal{T}f\|_{H^{s_0}(\Gamma)}}{\|\mathcal{T}f\|_{H^{s}(\Gamma)}}\right)^{\mu}, \text{ for all } h \in H_0^{s_0}(I). \tag{4.7}$$

Let s be any arbitrary real number such that $s > s_0$ (we will specify s later when needed). We now prove that this equality implies $[s_j(i_2)]^{1-\mu} \le C_s[s_j(i_1)]^{\mu}$ for all j, where i_1, i_2 are the natural embeddings $i_1 : H^s(\Gamma) \hookrightarrow H^{s_0}(\Gamma)$ and $i_2 : H_0^{s_1}(I) \hookrightarrow L^2(I)$. Indeed, let $\{h_j\}_j$ be the orthonormal basis of $H_0^{s_1}(I)$ such that

$$\|h_j\|_{L^2(I)} = s_j(i_2)\|h_j\|_{H^s(I)}.$$

Let $\alpha = (\alpha_1, ..., \alpha_j) \in \mathbb{R}^j$ and $h = \sum_{k\le j} \alpha_k h_k$. Since $\{h_j\}_j$ are also orthogonal in $L^2(I)$,

$$\begin{aligned} \|h\|^2_{L^2(I)} &= \sum_{k\le j} |\alpha_k|^2 \|h_k\|^2_{L^2(I)} = \sum_{k\le j} |\alpha_k|^2 s_k^2(i_2)\|h_k\|^2_{H^{s_1}(I)} \\ &\ge s_j^2(i_2) \sum_{k\le j} |\alpha_k|^2 \|h_k\|^2_{H^{s_1}(I)} = s_j^2(i_2)\|h\|^2_{H^{s_1}(I)}. \end{aligned}$$

That is,

$$\|h\|_{L^2(I)} \ge s_j(i_2)\|h\|_{H^{s_1}(I)}. \tag{4.8}$$

Now, consider $\{g_j\}_j$ to be the orthonormal basis of $H^s(\Gamma)$ such that

$$\|g_j\|_{H^{s_0}(\Gamma)} = s_j(i_1)\|g_j\|_{H^s(\Gamma)}.$$

Fixing j, we can choose $\alpha = (\alpha_1, ..., \alpha_j) \neq 0$ such that, for $h = \sum_{k\leq j} \alpha_k h_k$ and $f(x) = f_0(x')h(x_n)$, $\mathcal{T}(f)$ belongs to the orthogonal complement of $span\{g_1, ..., g_{j-1}\}$. Since the embedding $i_1 : H^s(\Gamma) \to H^{s_0}(\Gamma)$ has dense range, $\{g_k\}_k$ is also an orthogonal basis of $H^{s_0}(\Gamma)$. One then obtains $\mathcal{T}f = \sum_{k\geq j} \beta_k g_k$ in both $H^{s_0}(\Gamma)$ and $H^s(\Gamma)$. Therefore,

$$\|\mathcal{T}f\|^2_{H^{s_0}(\Gamma)} = \sum_{k\geq j} \beta_k^2 \|g_k\|^2_{H^{s_0}(\Gamma)} = \sum_{k\geq j} \beta_k^2 s_k^2(i_1) \|g_k\|^2_{H^s(\Gamma)} \leq s_j^2(i_1) \|\mathcal{T}f\|^2_{H^s(\Gamma)}.$$

That is,

$$\|\mathcal{T}f\|_{H^{s_0}(\Gamma)} \leq s_j(i_1) \|\mathcal{T}f\|_{H^s(\Gamma)}. \tag{4.9}$$

Combining (4.7), (4.8), and (4.9), one arrives at $[s_j(i_2)]^{1-\mu} \leq C_s [s_j(i_1)]^\mu$, where C_s is a constant independent of j. Due to Lemma C.3, this implies $[j^{-s_1}]^{(1-\mu)} \leq C_s [j^{\frac{s_0-s}{n}}]^\mu$. Choosing s big enough such that $s_1(1-\mu) < \frac{(s-s_0)\mu}{n}$, we have a contradiction by letting $j \to \infty$. The proof is completed.

D. An auxiliary result

Proposition D.1 *Assume that X, Y, Z are Banach spaces and Z is continuously imbedded into Y. Let $T : X \longrightarrow Y$ be a bounded linear operator such that $T(X) \subset Z$. Then $T : X \longrightarrow Z$ is also bounded.*

Proof Due to the closed graph theorem [52], if it is sufficient to prove that T has closed graph. That is, if $\{x_k\}_k \subset X$ such that $x_k \to x$ in X and $Tx_k \to z$ in Z, then $Tx = z$. In fact, since the imbedding $Z \to Y$ is continuous and $Tx_k \to z$ in Z, we have $Tx_k \to z$ in Y. On the other hand, since $x_k \to x$ in X and $T : X \longrightarrow Y$ is continuous, we have $Tx_k \longrightarrow Tx$ in Y. Due to the uniqueness of limit of the sequence $\{Tx_k\}_k$ in Y, we have $Tx = z$. This finishes the proof. ∎

CHAPTER V

SPEED DETERMINATION [1]

In this chapter, we investigate the problem of determining the ultrasound speed $c(x)$ from the TAT data g. Some numerical experiments have shown that it could be possible to recover the speed $c(x)$, simultaneously with the initial perturbation $f(x)$, from measured data g. However, no theoretical analysis has been done. We present here some initial results on this issue. We first show that the (unknown) constant speed can be uniquely determined by the TAT data. By using the range description for the wave operator, we prove a weak local uniqueness for variable speed. We then establish a necessary condition for the function $h(x) = c_1^{-2}(x) - c_2^{-2}(x)$ if c_1 and c_2 provide the same TAT data. Finally, we characterize the kernel of the linearized operator. In particular, we prove that the linearized operator, evaluated at the constant background speed, is injective in dimension one.

A. Introduction

Let us recall the commonly accepted model of TAT:

$$\begin{cases} u_{tt}(x,t) - c^2(x) \bigtriangleup u(x,t) = 0, \quad x \in \mathbb{R}^n, \quad t \geq 0, \\ u(x,0) = f(x), \quad u_t(x,0) = 0, \\ u(y,t) = g(y,t), y \in S, t \geq 0. \end{cases} \tag{5.1}$$

Here $c(x)$ is the the ultrasound speed at location x, $g(y,t)$ is the pressure measured at location $y \in S$ and time t, and $f(x)$ is the initial perturbation at location x.

---

[1] Part of this chapter is reprinted with permission from "Reconstruction and time reversal in thermoacoustic tomography in acoustically homogeneous and inhomogeneous media", by Yulia Hristova, Peter Kuchment, and Linh Nguyen, Inverse Problems 24 (5), 2008, 055006, doi: 10.1088/0266-5611/24/5/055006. Copyright ©2008 by IOP Publishing LTD.

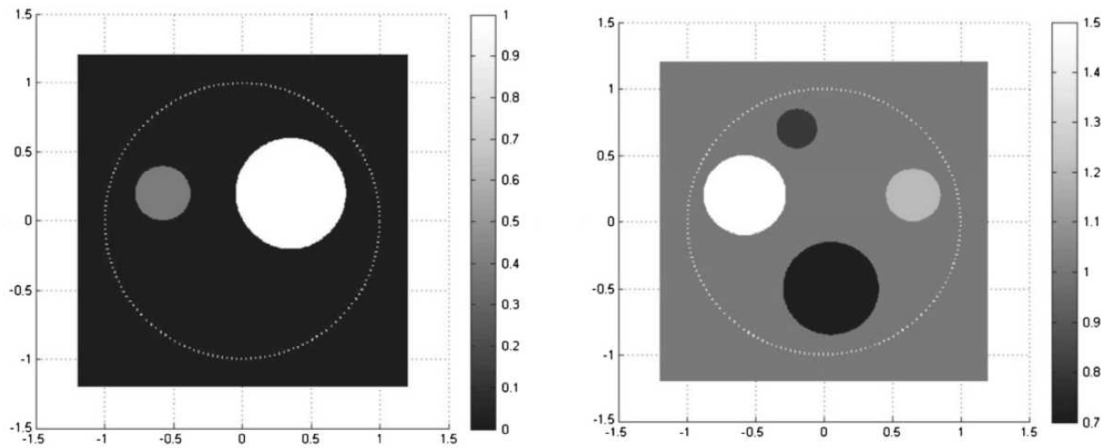

Fig. 3. The phantom and the sound speed density plot.

Most work in TAT relies on the assumption that the sound speed is known, while it is often not. To overcome this, one either tries to recover somehow the speed (e.g., using transmission ultrasound tomography [33, 32]), or assumes some (usually constant) speed. As it has been noted in previous studies (e.g., [33, 32, 69]), it is easy to conclude that using an incorrect sound speed deteriorates both the amplitudes, as well as locations of the features (e.g., of interfaces) of the image. The example shown here is provided to confirm this. In Fig. 3, one sees the speed map $c(x)$ and the phantom $f(x)$. Fig. 4 shows the reconstructions using the true speed and the average speed, by the time reversal method (see, e.g., [31]). One can easily see both aforementioned types of deterioration in the case of the average speed.

Therefore, it would be extremely valuable to be able to recover the speed map from the TAT data. A numerical simultaneous reconstruction of the speed and the image was successfully attempted on examples in [70]. It is, thus, natural to formulate the following problem:

Problem A.1 *Does the TAT data g uniquely determine $c(x)$ and $f(x)$? If not, to what extent it does?*

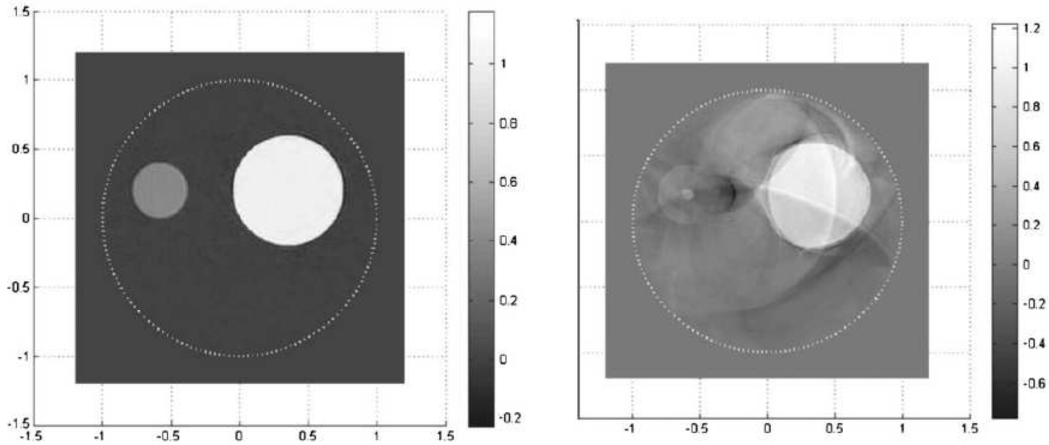

Fig. 4. The reconstruction with the correct speed map (left) and with the average speed (right). One observes both location shifts and amplitude deterioration of the reconstruction with the average speed.

Once the speed $c(x)$ is determined, the determination of f follows the problem of TAT with known ultrasound speed, which has been well studied (see, e.g., [20, 55]). Therefore, Problem A.1, indeed, boils down to the determination of the speed $c(x)$ from g.

We provide here a natural point of view to the Problem A.1. Consider the linear operator $L_{c(x)}$ that, given a sound speed $c(x)$, transforms the initial function $f(x)$ into the boundary TAT data $g(y,t)$. Then we would like to recover from the equation $L_{c(x)}f = g$ with a given g both the sound speed c and the image f. How can this be possible? A simple-minded objection is: given whatever c, one can solve this equation and find a solution f, and thus the equation does not carry any information about c. This argument, however, is correct only if the operators L_c are invertible, or "almost" invertible (i.e., the ranges of the operators are equal to almost the whole space of functions g). However, it is common in tomography that the range of L_c is very small, namely, of infinite co-dimension in natural spaces of functions g [43, 44].

Suppose that a range description for this transform is known (in appropriate spaces). A change of the sound speed leads to a "rotation" of the range. If the ranges rotate so much that their intersections for different sound speeds are only at the zero function, then given the data g, one could find out to the range of which operator $L_{c(x)}$ it belongs and thus determine $c(x)$. This would allow one to recover the sound speed, and then, using any of the known methods (e.g., [31]), to reconstruct the image f. One can notice the similarity with the SPECT (single photon emission tomography), where one faces the still not satisfactory resolved problem of the simultaneous recovery of the unknown attenuation coefficient (kind of an analog of the speed in TAT) and of the unknown source intensity distribution function [29, 53, 43, 44].

In this chapter, we present some initial results on Problem A.1. We prove that the unknown constant speed is uniquely determined by the TAT data in Section B. In Section C, we prove some local uniqueness for the variable speed from the TAT data. We characterize the difference $h(x) = c_1^{-2}(x) - c_2^{-2}(x)$ for the speeds c_1 and c_2 that provide the same TAT data g in Section D. Finally, some analysis for the linearized operator is provided in Section E.

B. Uniqueness of the constant speed

We first look at the simplest question: knowing in advance that the sound speed is constant, can one recover it from the TAT data? We will show the positive answer in odd dimensions [2] $n > 1$. Let us assume that S is the unit sphere and the initial perturbation $f(x)$ is compactly supported in the open unit ball B, whose boundary is S. This assumption implies that the support of $f(x)$ does not reach S. In this case, the finiteness of the speed of propagation shows that the boundary data $g(y, t)$

[2]The case $n = 1$ also follows from the same (even simpler) argument.

is equal to zero for small t. Let us thus define the first time $t_0 > 0$ when the signal reaches the boundary as follows:

$$t_0 = \inf\{t > 0 | \text{ there exists } y \in S \text{ such that } g(y,t) \neq 0\}. \tag{5.2}$$

Since the speed of sound is constant and the dimension is odd, the Huygens' principle holds. Thus, the data g will vanish for large values of time t. We can define the last time $T_0 > 0$ when the signal is detected:

$$T_0 = \sup\{t > 0 | \text{ there exists } y \in S \text{ such that } g(y,t) \neq 0\}. \tag{5.3}$$

The following result resolves the question of recovering the constant sound speed:

Theorem B.1 *Let, as above, the initial perturbation f be supported inside the open ball B and the numbers $t_0, T_0 > 0$ be defined as in (5.2)-(5.3). Then the sound speed c satisfies the equality*

$$c = \frac{2}{t_0 + T_0} \tag{5.4}$$

and thus is uniquely determined by the TAT data g.

Proof Let us recall that pressure wave $u(x,t)$ solves the following wave equation problem in $\mathbb{R}^n$:

$$\begin{cases} u_{tt}(x,t) - c^2 \Delta u(x,t) = 0, \quad t > 0, \\ u(x,0) = f(x), \; u_t(x,0) = 0, \forall x \in \mathbb{R}^n. \end{cases}$$

The standard Kirchhoff-Poisson solution formulas for the wave equation (see [11] or [15, p.77]) imply the representation:

$$\begin{aligned} g(y,t) &= c_n \left[\left(\frac{\partial}{\partial t} \frac{1}{t} \right)^{\frac{n-1}{2}} \mathcal{R}f \right](y, ct) \\ &= 2^{\frac{n-1}{2}} c_n t \left[\left(\frac{1}{2t} \frac{\partial}{\partial t} \right)^{\frac{n-1}{2}} \frac{1}{t} \mathcal{R}f \right](y, ct) \end{aligned}$$

$$= \quad 2^{\frac{n-1}{2}} c_n t \left[\left(\frac{\partial}{\partial(t^2)} \right)^{\frac{n-1}{2}} \frac{1}{t} \mathcal{R} f \right] (y, ct), \quad y \in S, \quad t > 0. \tag{5.5}$$

Here, $\mathcal{R}$ is the spherical Radon transform defined in Chapter I. The assumptions of the theorem imply that t_0 is the largest value of t such that $g(y,t) = 0$ for all $y \in S$ and $t < t_0$.

Lemma B.2 *The equality $g(y,t) = 0$ being satisfied for all $y \in S$ and $t < t_0$ is equivalent to the equality $\mathcal{R}(f)(y, ct) = 0$ being satisfied for all these values.*

Indeed, due to the formula (5.5), if $\mathcal{R}(f)(y, ct) = 0$ for all $y \in S$ and $t < t_0$, this implies $g(y,t) = 0$ in the same region. Let us now prove the converse statement. We recall that the assumption of the support of f not reaching S implies that $\mathcal{R}f(y,t)$ vanishes for sufficiently small values of t. On the other hand, the last formula in (5.5) implies that

$$\left[\left(\frac{\partial}{\partial(t^2)} \right)^{\frac{n-1}{2}} \frac{1}{t} \mathcal{R} f \right] (y, ct) = 0$$

for all $y \in S$ and $t < t_0$. These two facts and integration with respect to t^2 imply that $\mathcal{R}(f)(y, ct) = 0$ for all $y \in S$ and $t < t_0$. This proves the lemma.

The same argument (with $\mathcal{R}(f) = 0$ for small values of t being replaced by the Huygens' principle), implies the following statement:

Lemma B.3 *The equality $g(y,t) = 0$ being satisfied for all $y \in S$ and $t > T_0$ is equivalent to the equality $\mathcal{R}(f)(y, ct) = 0$ being satisfied for all these values.*

The above lemmas lead to an alternative description of the numbers t_0 and T_0 in terms of the spherical mean transform of f rather than the TAT data g:

Corrolary B.4 *The following formulas hold true:*

$$\begin{aligned} t_0 &= \tfrac{1}{c} \sup\{t : \mathcal{R}(f)(x,s) = 0, \ \textit{for all } x \in S, s \leq t\}, \\ T_0 &= \tfrac{1}{c} \inf\{t : \mathcal{R}(f)(x,s) = 0, \ \textit{for all } x \in S, s \geq t\}. \end{aligned} \tag{5.6}$$

We will now use relations (5.6) to evaluate c in terms of t_0, T_0.

Let us define two quantities related to the support of f:

$$\sigma_0 = \min\{|x-y| : \quad x \in supp(f), y \in S\},$$

$$\Sigma_0 = \max\{|x-y| : \quad x \in supp(f), y \in S\}.$$

The following lemma clearly finishes the proof of the theorem:

Lemma B.5 *The numbers σ_0, Σ_0 satisfy the following properties:*

1. $\sigma_0 + \Sigma_0 = 2$.

2.
$$\sigma_0 = \sup\{t : \mathcal{R}f(x,s) = 0, \ \text{for all } x \in S, s \leq t\},$$
$$\Sigma_0 = \inf\{t : \mathcal{R}f(x,s) = 0, \ \text{for all } x \in S, s \geq t\}.$$

3. $\sigma_0 = ct_0, \Sigma_0 = cT_0$

Proof Let us introduce the radius of the smallest ball centered at the origin that encloses the support of f:

$$r_0 = \sup\{|x| \mid \quad f(x) \neq 0\} < 1.$$

Then it is clear that $\sigma_0 = 1 - r_0$ and $\Sigma_0 = 1 + r_0$. This implies the first statement of the lemma. The third statement follows from the second one and the formulas (5.6).

It remains to prove the second statement of the lemma. In its generality, it follows from the non-trivial local uniqueness theorem for the spherical transform [42, Theorem 4]. This theorem, in particular, requires the support of f to be strictly inside S, which is one of our assumptions. However, in TAT applications, the initial function f is non-negative. Knowing this additional information, the statement is very easy to prove. Indeed, the following inequalities are obvious:

$$\sigma_0 \leq \sup\{t : \mathcal{R}f(x,s) = 0, \text{ for all } x \in S, s \leq t\}, \tag{5.7}$$

$$\Sigma_0 \geq \inf\{t : \mathcal{R}f(x,s) = 0, \text{ for all } x \in S, s \geq t\}. \tag{5.8}$$

It remains to prove that the inequalities

$$\sigma_0 < \sup\{t : \mathcal{R}f(x,s) = 0, \text{ for all } x \in S, s \leq t\} \tag{5.9}$$

and

$$\Sigma_0 > \inf\{t : \mathcal{R}f(x,s) = 0, \text{ for all } x \in S, s \geq t\} \tag{5.10}$$

are impossible. Let us show this for the first of these inequalities. According to the definition of σ_0, there is a point $x_0 \in \mathit{supp}(f)$ such that the distance from x_0 to S is equal to σ_0. Let y_0 be the closest point to x_0 on S. Then Fubini theorem implies that for an arbitrarily small $\epsilon > 0$ there is a number $\sigma_0 < s < \sigma_0 + \epsilon$ such that the sphere $\{x \,|\, |x - y_0| = s\}$ intersects the support of f over a set of a positive surface measure. Due to the positivity of f, this implies that $\mathcal{R}(f)(y_0, s) \neq 0$. This proves the impossibility of the inequality (5.9). Impossibility of (5.10) is proven analogously. This finishes the proof of the lemma and thus of the theorem. ∎

Remark B.6 *Assume that $f \in C_0^\infty(\overline{B})$ and $\mathit{supp}(f) \cap S \neq \emptyset$. An argument similar to above shows that the first arrival time $t_0 = 0$ and the last detected time $T_0 = \frac{2}{c}$. Hence, Theorem B.1 is still true for this case.*

C. Range conditions and speed determination

Range conditions for the case of a constant sound speed are presented in Chapter III. They contain two types of restrictions on the TAT data g: one concerns the smoothness and support of g and the other involves orthogonality of g to certain functions.

We provide here without a proof an analog of the orthogonality condition for the case of a variable sound speed $c(x)$. Let us assume that $c(x)$ is smooth, stabilizes to a constant $c > 0$ at large distances (and thus the support of $c(x) - c$ is compact), and satisfies the non-trapping condition (see Definition C.3). The condition described in the next theorem is necessary for a function g to belong to the range of $L_{c(x)}$, while we cannot claim its sufficiency, as it was done for the constant speed case in Chapter III. In fact, sufficiency would require some additional constrain(s).

Let us denote by B the domain bounded by the observation surface [3] S. We consider, in the Hilbert space $L^2(c^{-2}(x), B)$, the operator $A := -c^2(x)\Delta$ with zero Dirichlet boundary conditions on S and the natural domain $H^2(B) \bigcap H_0^1(B)$. It is a positive self-adjoint operator. The following result is analogous to the orthogonality condition in Chapter III:

Theorem C.1 *Let $\{\lambda_k^2\}_{k=1}^{\infty}$ be the spectrum of A and $\{\psi_k\}_k$ be the corresponding basis of orthonormal eigenfunctions in $L^2(c^{-2}(x), B)$. If $u(x,t)$ solves the problem*

$$\begin{cases} u_{tt}(x,t) - c^2(x)\Delta u(x,t) = 0, \ \textit{for all } x \in \mathrm{I\!R}^n, t > 0 \\ u(x,0) = f(x) \in C_0^\infty(\mathrm{I\!R}^n), \quad u_t(x,0) = 0, \\ u|_{S \times \mathrm{I\!R}_+} = g, \end{cases}$$

then the function $g(x,t)$ on $S \times \mathrm{I\!R}_+$ satisfies, for any k, the condition

$$\int\limits_0^\infty \left(\int\limits_S g(x,t) \frac{\partial \psi_k}{\partial \nu} \right) \cos(\lambda_k t) dt = 0, \tag{5.11}$$

where ν is the exterior unit normal vector to S.

[3] We do not assume special geometries of S in the rest of this Chapter.

The proof of this result is very similar to the one given in [4] for the constant speed case. Condition (5.11) is not sufficient for g to be in the range of L_c. In particular, the range description in the constant speed case also needs the support and smooth condition (see Chapter III).

We will make some observations concerning the usage of the orthogonality condition provided in Theorem C.1 for the speed determination. The conclusion we reach here is that this condition alone does not guarantee uniqueness even for constant speeds. However, it leads to a (rather weak) local uniqueness result.

Lemma C.2 *The orthogonality condition (5.11) alone does not determine a constant speed c uniquely.*

The following example in $3D$ proves the non-uniqueness. Let f be a radial function supported inside the unit ball B and u solve the problem:

$$\begin{cases} u_{tt}(x,t) - \triangle u(x,t) = 0, \forall x \in \mathbb{R}^n, t > 0, \\ u(x,0) = f(x), \quad u_t(x,0) = 0. \end{cases} \tag{5.12}$$

Let $g = u|_{S\times\mathbb{R}^+}$. Then the range description (5.11) says

$$\int\limits_0^\infty \int\limits_S g(x,t)\phi_k(x)\cos(\lambda_k t)dt = 0,$$

when ϕ_k is any spherical harmonic of degree k, and λ_k is a zero of the spherical Bessel function $j_{k+(n-2)/2}$. Since f is radial, so is g, and thus the condition need to be checked on radial functions (i.e., for $k = 0$) only:

$$\int\limits_0^\infty \int\limits_S g(x,t)\cos(\lambda t)dt = 0$$

for λ being a zero of $j_{(n-2)/2}$.

Let us choose $n = 3$, and thus $j_{(n-2)/2}(\lambda) = j_{1/2}(\lambda) = \frac{\sin(\lambda)}{\lambda}$. We then have

$\lambda = l\pi$. Therefore, if λ is a zero of $j_{(n-2)/2}$ then so is $m\lambda$ for any natural m. Now one can see that g satisfies the orthogonality condition for the speed $c_1 = mc$ as well, which provides the needed counterexample.

It is clear that the counterexample would not work for the coefficient m being close to 1. This observation will lead to a (weak) local uniqueness result presented in the next theorem. First of all, we assume again that the speed $c(x) > 0$ is smooth, stabilizes to a constant at infinity, and satisfies the non-trapping condition, which is defined as follows:

Definition C.3 *Consider the Hamiltonian system in* $\mathbb{R}^{2n}_{x,\xi}$ *with the Hamiltonian* $H = \frac{c^2(x)}{2}|\xi|^2$*:*

$$\begin{cases} x'_s = \frac{\partial H}{\partial \xi} = c^2(x)\xi \\ \xi'_s = -\frac{\partial H}{\partial x} = -\frac{1}{2}\nabla\left(c^2(x)\right)|\xi|^2 \\ x|_{s=0} = x_0, \xi|_{s=0} = \xi_0. \end{cases}$$

The solutions of this system are called bicharacteristics and their projections into $\mathbb{R}^n_x$ *are rays. We will say that the* **non-trapping condition** *holds, if all rays (with* $\xi_0 \neq 0$*) tend to infinity when* $t \to \infty$*.*

Theorem C.4 *Let the dimension* $n > 1$ *be odd,* B *be a bounded domain in* $\mathbb{R}^n$ *with a smooth boundary* S*, and* g *be a non-zero function on* $S \times \mathbb{R}^+$ *such that the problem*

$$\begin{cases} u_{tt} - c^2(x)\Delta u = 0, \quad x \in \mathbb{R}^n, t > 0, \\ u(x,0) \in C_0^\infty(\mathbb{R}^n), \quad u_t(x,0) = 0 \end{cases} \tag{5.13}$$

has a solution u *satisfying* $u|_{S\times\mathbb{R}_+} = g$*.*

Then there exists $\varepsilon_0 > 0$ such that for all $0 < |\varepsilon| < \varepsilon_0$, the problem

$$\begin{cases} u_{tt} - (1+\varepsilon)^2 c^2(x)\Delta u = 0, \quad x \in \mathbb{R}^n, t > 0, \\ u(x,0) \in C_0^\infty(\mathbb{R}^n), \quad u_t(x,0) = 0 \\ u|_{S\times\mathbb{R}_+} = g \end{cases}$$

has no solution.

Proof Let λ_k^2 be an eigenvalue of the operator $A = -c^2(x)\Delta$ in B with zero Dirichlet conditions on S and ψ_k be an associated eigenfunction. Consider

$$G(t) = \int\limits_S g(x,t)\frac{\partial\psi_k}{\partial\nu}(x).$$

Notice that since g is not identically equal to zero and due to completeness of functions ψ_k in $L^2(c^{-2}(x), B)$, we can pick a value of k for which G is not identically equal to zero. We also have

$$|G(t)| \leq \left\|\frac{\partial\psi_k}{\partial\nu}\right\|_{L^2(S)} \|g(\cdot,t)\|_{L^2(S)} \leq C\left\|\frac{\partial\psi_k}{\partial\nu}\right\|_{L^2(S)} \|u(\cdot,t)\|_{H^1(B)}.$$

Due to the non-trapping condition imposed on $c(x)$ and dimension being odd, $\|u(\cdot,t)\|_{H^1(B)}$ decays exponentially at infinity, and hence so does $G(t)$. We extend G to an even function with respect to t and consider

$$\hat{G}(\xi) = \int\limits_{-\infty}^{\infty}\left(\int\limits_S g(x,t)\frac{\partial\psi_k}{\partial\nu}(x)\right)e^{i\xi t}dt.$$

Due to the exponential decay of G, this function is analytic in the strip $|Im(z)| < \delta$ for small enough $\delta > 0$. Since G, and hence $\hat{G}$, is not identically equal to zero, zeros of $\hat{G}$ are isolated. Moreover, the orthogonality condition (5.11) implies that λ_k is a zero of $\hat{G}$. Hence, there exists $\varepsilon_0 > 0$ such that $(1+\varepsilon)\lambda_k$ is not a zero of $\hat{G}$ for any ε such that $0 < |\varepsilon| < \varepsilon_0$.

On the other hand, suppose $g = v|_{S\times\mathrm{IR}^+}$ for some v solving the problem

$$\begin{cases} v_{tt}(x,t) - \rho^2 c^2(x)\Delta v(x,t) = 0, \quad x \in \mathrm{IR}^n, t > 0, \\ v(x,0) \in C_0^\infty(\mathrm{IR}^n), \quad v_t(x,0) = 0. \end{cases} \tag{5.14}$$

Applying the range condition (5.11) with $c(x)$ replaced by $\rho c(x)$, and hence the zero λ_k replaced by $\rho\lambda_k$, we see that then $\hat{G}(\rho\lambda_k) = 0$. On the other hand, since $\rho \in (1-\varepsilon_0, 1+\varepsilon_0)$, this cannot happen. This finishes the proof.

D. Characterization of the non-uniqueness

Let us reformulate Problem A.1 as follows:

Problem D.1 *Let (c_1, f_1) and (c_2, f_2) be two pairs of speed and initial perturbation. Let g_1, g_2 be the corresponding TAT data, defined by equation (5.1). Assuming that $g_1 = g_2$, what can we say about $h(x) = c_1(x) - c_2(x)$?*

If we are able to show $h(x) = 0$ for all $x \in \mathrm{IR}^n$, we conclude that the TAT data uniquely determines the speed $c(x)$. We present here a partial result for problem (D.1):

Theorem D.2 *Assume (c_1, f_1) and (c_2, f_2) are such that $g_1 = g_2$, and c_2 is constant. If $c_2(x) - c_1, f_2 - f_1 \in C_0^\infty(\overline{B})$, then $h = c_1^{-2}(x) - c_2^{-2}$ satisfies $h = \Delta V$, for some function $U \in C_0^\infty(\overline{B})$.*

In order to prove this theorem, we need an auxiliary result:

Lemma D.3 *Let f be a nonnegative nonzero function and u solve the wave equation:*

$$\begin{cases} u_{tt}(x,t) - \triangle u(x,t) = 0, \; x \in \mathrm{IR}^n, \; t \geq 0, \\ u(x,0) = f(x), \quad u_t(x,0) = 0. \end{cases}$$

Denote by $\hat{u}(x,\lambda)$ the Fourier transform of f with respect to t. There exist a nonzero constant C such that $\lim_{\lambda\to 0^+} \frac{\hat{u}(x,\lambda)}{\lambda^{n-1}} = C$ for all $x \in \mathbb{R}^n$. Moreover, the convergence is uniform on any compact sets.

Proof If n is odd, let us recall the Kirchhoff-Poisson solution formula for the wave equation (see [11] or [15, p.77]):

$$u(x,t) = c_n \left[\left(\frac{\partial}{\partial t}\frac{1}{t}\right)^{\frac{n-1}{2}} \mathcal{R}(f)\right](x,t), \quad x \in \mathbb{R}^n, \quad t > 0.$$

Thus,

$$\begin{aligned}
\hat{u}(x,\lambda) &= 2c_n \int\limits_{\mathbb{R}_+} \left[\left(\frac{\partial}{\partial t}\frac{1}{t}\right)^{\frac{n-1}{2}} \mathcal{R}(f)\right](x,t)\cos(\lambda t)dt \\
&= (-1)^{\frac{n-1}{2}} c_n \int\limits_{\mathbb{R}_+} \mathcal{R}(f)(x,t)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-1}{2}} \cos(\lambda t)dt \\
&= (-1)^{\frac{n-3}{2}} c_n \lambda \int\limits_{\mathbb{R}_+} \mathcal{R}(f)(x,t)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-1}{2}} \left(\frac{\sin(\lambda t)}{t}\right) dt.
\end{aligned}$$

Due to (2.48) and (2.3) (see Chapter II):

$$\begin{aligned}
c_n \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-1}{2}} \left(\frac{\sin(\lambda t)}{t}\right) &= \mathcal{I}G(t,\lambda) = \mathcal{I}\left[\frac{i}{4}\left(\frac{\lambda}{2\pi s}\right)^{\frac{n-2}{2}} H^{(1)}_{\frac{n-2}{2}}(\lambda t)\right] \\
&= \mathcal{I}\left[\frac{i}{4}\left(\frac{\lambda}{2\pi s}\right)^{\frac{n-2}{2}} \left(J_{\frac{n-2}{2}}(\lambda t) + iN_{\frac{n-2}{2}}(\lambda t)\right)\right] \\
&= \frac{1}{4}\left(\frac{\lambda}{2\pi s}\right)^{\frac{n-2}{2}} J_{\frac{n-2}{2}}(\lambda t) = c\lambda^{n-2}\frac{J_{\frac{n-2}{2}}(\lambda t)}{(\lambda t)^{n-2}}.
\end{aligned}$$

Here c is a nonzero constant and $J_{\frac{n-2}{2}}$ is the Bessel function of order $\frac{n-2}{2}$.

Therefore,

$$\hat{u}(x,\lambda) = c\lambda^{n-1} \int\limits_{\mathbb{R}_+} \mathcal{R}(f)(x,t)\frac{J_{\frac{n-2}{2}}(\lambda t)}{(\lambda t)^{n-2}} dt. \tag{5.15}$$

Let us recall the normalized Bessel function

$$j_{\frac{n-2}{2}}(s) = s^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(s).$$

We derive from (5.15) the equation:

$$\lim_{\lambda\to 0} \frac{\hat{u}(x,\lambda)}{\lambda^{n-1}} = c j_{\frac{n-2}{2}}(0) \int\limits_{\mathbb{R}_+} \mathcal{R}(f)(x,t)dt = c j_{\frac{n-2}{2}}(0)\|f\|_{L^1(\mathbb{R}^n)}.$$

Since $j_{\frac{n-2}{2}}(0) \neq 0$, the limits is also nonzero. The uniform convergence is quite obvious since f is smooth and compactly supported. This finishes the proof for odd n. Similar argument provides the proof for even n. ∎

Proof of Theorem D.2 Without loss of generality, we assume that $c_1 = 1$. Taking the Fourier transform of the equation (5.1), we obtain:

$$\begin{cases} \Delta \hat{u}_1(x,\lambda) + c_1^{-2}(x)\lambda^2 \hat{u}_1(x,\lambda) = 0, \\ \Delta \hat{u}_2(x,\lambda) + \lambda^2 \hat{u}_2(x,\lambda) = 0, \\ \hat{u}_1(y,\lambda) = \hat{u}_2(y,\lambda), \partial_\nu \hat{u}_1(y,\lambda) = \partial_\nu \hat{u}_2(y,\lambda) = 0,\ y \in \partial\Omega. \end{cases}$$

Let $v_i(x,\lambda) = \partial_\lambda^{(n-1)} \hat{u}_i(x,\lambda)$. Due to Lemma D.3, we obtain $v_2(x,0)$ is equal to a nonzero constant. The above equation implies:

$$\begin{cases} \Delta v_1(x,\lambda) + \lambda^2 c_1^{-2}(x) v_1(x,\lambda) = 0, \\ \Delta v_2(x,\lambda) + \lambda^2 v_2(x,\lambda) = 0, \\ v_1(y,\lambda) = v_2(y,\lambda), \partial_\nu v_1(y,\lambda) = \partial_\nu v_2(y,\lambda) = 0,\ y \in \partial\Omega. \end{cases} \tag{5.16}$$

Multiplying the first equation by v_2 and the second one by v_1, and then subtracting them, we arrive at

$$\nabla\left[v_2(x)\nabla v_1(x) - v_1(x)\nabla v_2(x)\right] + \lambda^2 h(x) v_1(x) v_2(x) = 0. \tag{5.17}$$

Since $v_2(x,0)$ is a positive constant on $\mathbb{R}^n$ and $v_2(.,\lambda) \to v_2(x,0)$ uniformly on $\overline{B}$ (Lemma D.3), $v_2(x,\lambda) \neq 0$ on $\overline{B}$ for small enough λ. We obtain the following elliptic equation:

$$\begin{cases} \nabla\left(\rho_\lambda(x)\nabla U_\lambda(x)\right) + \lambda^2\rho_\lambda(x)h(x)U_\lambda(x) = 0, \\ U|_{\partial\Omega} = 1, \frac{\partial U}{\partial\nu}|_{\partial\Omega} = 0, \end{cases} \tag{5.18}$$

where $U_\lambda(x) = \frac{v_1(x,\lambda)}{v_2(x,\lambda)}$, and $\rho_\lambda(x) = v_2^2(x,\lambda)$.

We now consider the series expressions:

$$\rho_\lambda = \rho_0 + \lambda^2\rho_1 + \lambda^4\rho_2 + ...,$$
$$U_\lambda = U_0 + \lambda^2 U_1 + \lambda^4 U_2 + ...,$$

where $U_0 = 1$, and $U_j = \frac{\partial U_j}{\partial_\nu} = 0$ on $\partial\Omega$ for all $j \geq 1$.

Identifying the terms with respect to λ^2 in equation (5.18), we obtain

$$\begin{cases} \nabla\left(\rho_0(x)\nabla U_1(x)\right) + \rho_0(x)h(x)U_0 = 0, \\ U_1|_{\partial\Omega} = \frac{\partial U_1}{\partial\nu}|_{\partial\Omega} = 0, \end{cases}$$

Since ρ_0 is constant and $U_0 = 1$, we obtain $h(x) = -\Delta U_1(x)$. The theorem is proved, with $V = -U_1$. ∎

E. The linearized problem

Let us define the operator $\mathcal{L}$ which sends $(c^2(x), f(x))$ to the TAT data g:

$$\mathcal{L}(c^2(x), f(x)) = g.$$

The problem of speed determination is to consider the injectivity of $\mathcal{L}$. One possible approach is to study the linearized operator of $\mathcal{L}$. Let us formally linearize the operator $\mathcal{L}$ at the background speed $c(x)$ and the given initial perturbation $f(x)$.

Letting $h, k \in C_0^\infty(\overline{B})$, we consider the nonhomogeneous wave equation:

$$\begin{cases} v_{tt}(x,t) - c^2(x) \bigtriangleup v(x,t) = h(x)u_{tt}(x,t), \quad \forall x \in \mathbb{R}^n, \quad t > 0, \\ v(x,0) = k(x), \quad v_t(x,0) = 0, \quad \forall x \in \mathbb{R}^n. \end{cases}$$

where u is the solution of the homogeneous equation (5.1).

Then the linearized operator $\Lambda_{(c^2(x),f(x))}$ of $\mathcal{L}$ at $(c^2(x), f(x))$ is defined by:

$$\Lambda_{(c^2(x),f(x))}(h,k) = v|_{S\times\mathbb{R}_+}.$$

If $\Lambda_{(c(x),f(x))}(h,k) = 0$ then v satisfies the equation:

$$\begin{cases} v_{tt}(x,t) - c^2(x) \bigtriangleup v(x,t) = -h(x)u_{tt}(x,t), \quad \forall x \in \mathbb{R}^n, t > 0, \\ v(y,t) = 0, \quad \forall (y,t) \in S \times \mathbb{R}_+, \\ v(x,0) = k(x), \ v_t(x,0) = 0, \quad x \in \mathbb{R}^n. \end{cases} \tag{5.19}$$

The following result follows directly TAT problem with given speed:

Proposition E.1 *Let $h, k \in C_0^\infty(\overline{B})$ such that $\Lambda_{(c^2(x),f(x))}(h,k) = 0$. If $h = 0$ then $k = 0$.*

We prove here a result, similar to Theorem D.2 (for the nonlinear problem), which characterizes the kernel of operator $\Lambda_{(c^2(x),f(x))}$:

Theorem E.2 *Assume that $c = 1$ and $f \in C_0^\infty(\overline{B})$. Let $h, k \in C_0^\infty(\overline{B})$ such that $\Lambda_{(c^2(x),f(x))}(h,k) = 0$. Then $h = \Delta V$, for some function $V \in C_0^\infty(\overline{B})$.*

Lemma E.3 *Consider the equation (5.19). Let $(\lambda^2, \varphi_\lambda)$ be a pair of eigenvalue-eigenfunction of $-c^2(x)\Delta$ on B (without boundary condition). Then*

$$\int_B c^{-2}(x)h(x)\varphi_\lambda(x)\hat{u}(x,\lambda)dx = 0. \tag{5.20}$$

Proof Since h, k are supported inside $\overline{B}$, v satisfies the equation:

$$\begin{cases} v_{tt}(x,t) = \triangle v(x,t), \quad \forall x \in \mathbb{R}^n \setminus B, \quad t > 0, \\ v(y,t) = 0, \quad \forall y \in S, t \geq 0, \\ v(x,0) = 0, \quad v_t(x,0) = 0, \quad \forall x \in \mathbb{R}^n \setminus B \end{cases} \tag{5.21}$$

A simple energy argument then shows that $v \equiv 0$ on $(\mathbb{R}^n \setminus B) \times \mathbb{R}_+$. Let us extend v evenly with respect to t to $\mathbb{R}^n \times \mathbb{R}$. Then $v(x,t) = 0$ on $(\mathbb{R}^n \setminus \overline{B}) \times \mathbb{R}$, and v solves the wave equation

$$v_{tt}(x,t) - c^2(x) \triangle v(x,t) = h(x) u_{tt}(x,t), \quad \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}.$$

Multiplying the above equation by $\varphi_\lambda(x) e^{i\lambda t}$ and taking the integration on $B \times \mathbb{R}$, we obtain:

$$\int_B \int_{\mathbb{R}} \left[c^{-2}(x) v_{tt}(x,t) - \triangle v(x,t) \right] \varphi_\lambda(x) e^{i\lambda t} dt dx = \int_B c^{-2}(x) h(x) \varphi_\lambda(x) \hat{u}(x,\lambda) dx.$$

Taking integration by parts, one sees that the left hand side is zero. Thus,

$$\int_B c^{-2}(x) h(x) \varphi_\lambda(x) \hat{u}(x,\lambda) dx = 0.$$

∎

Proof of Theorem E.2 Since $c = 1$, equation (5.20) becomes

$$\int_B h(x) \varphi_\lambda(x) \hat{u}(x,\lambda) dx = 0, \tag{5.22}$$

for all φ_λ satisfying

$$-\Delta \varphi_\lambda(x) = \lambda^2 \varphi_\lambda(x), \text{ for all } x \in B. \tag{5.23}$$

We now prove that for any harmonic function $\varphi \in H^1(B)$,

$$\int_B h(x)\varphi(x)dx = 0, \tag{5.24}$$

Indeed, for each $\lambda > 0$, let φ_λ satisfy (5.23) and $\varphi_\lambda = \varphi$ on ∂B. We derive from (5.23) the equation:

$$\Delta(\varphi_\lambda - \varphi) + \lambda(\varphi_\lambda - \varphi) = -\lambda\varphi.$$

Multiply the equation by $(\varphi_\lambda - \varphi)$ and taking integration by parts, we obtain:

$$\|\nabla(\varphi_\lambda - \varphi)\|^2_{L^2(B)} - \lambda\|\varphi_\lambda - \varphi\|^2_{L^2(B)} = \lambda \int_{L^2(B)} \varphi(x)(\varphi_\lambda - \varphi)(x)dx. \tag{5.25}$$

Let us recall the Poincare's inequality

$$\|\varphi_\lambda - \varphi\|_{L^2(B)} \leq C\|\nabla(\varphi_\lambda - \varphi)\|_{L^2(B)}.$$

Choosing λ in (5.25) small enough, and applying the Hölder inequality, we obtain:

$$\|\varphi_\lambda - \varphi\|_{L^2(B)} \leq C\lambda\|\varphi\|_{L^2(B)}.$$

This implies

$$\|\varphi_\lambda - \varphi\|_{L^2(B)} \to 0, \quad \lambda \to 0. \tag{5.26}$$

From equation (5.22), we arrive at

$$\lim_{\lambda \to 0} \int_B h(x)\varphi_\lambda(x)\frac{\hat{u}(x,\lambda)}{\lambda^{n-1}}dx = 0.$$

The convergence (5.26) and that $\frac{\hat{u}(x,\lambda)}{\lambda^{n-1}}$ converges uniformly to a nonzero constant (Lemma D.3) then conclude (5.24).

Let V solve the equation

$$\begin{cases} \Delta V(x) = h(x), & x \in B, \\ V(y) = 0, & y \in \partial S. \end{cases} \tag{5.27}$$

We now that $\frac{\partial V}{\partial \nu}(y) = 0$ for all $y \in S$. For any function $\psi \in H^{1/2}(\partial\Omega)$, there is a harmonic function $\varphi \in H^1(B)$, such that $\varphi|_{\partial B} = \psi$. Multiplying equation (5.27) by φ, and taking the integration by parts, we obtain:

$$\int_{\partial B} \frac{\partial V(y)}{\partial \nu} \psi(y) d\sigma(y) = \int_B h(x)\varphi(x)dx.$$

Due to (5.24), the right hand side is zero, and so is the left hand side. Since it is true for all $\psi \in H^{1/2}(B)$, we obtain $\frac{\partial V(y)}{\partial \nu} = 0$ for all $y \in \partial B$.

Finally, due to the equation (5.27) and $h \in C_0^\infty(h)$, we conclude that $V \in C_0^\infty(\overline{B})$.∎

Remark E.4 *If $n \geq 2$, (5.15) gives:*

$$\hat{u}(x, \lambda) = c\lambda^{n-2} \int_{\mathbb{R}_+} \mathcal{R}(f)(x,t) \frac{J_{\frac{n-2}{2}}(\lambda t)}{(\lambda t)^{n-2}} dt.$$

Choosing $\varphi_\lambda(x) = J(\lambda|x - x_0|)$ in (5.22), we arrive at

$$\int_\Omega h(x) J(\lambda|x - x_0|) \int_{\mathbb{R}_+} \mathcal{R}(f)(x,t) \frac{J_{\frac{n-2}{2}}(\lambda t)}{(\lambda t)^{n-2}} dt = 0.$$

By exploiting this identity further, we hope to obtain some injectivity result in the future for $n \geq 2$. In the meanwhile, we provide below the injectivity of $\Lambda_{(1,f(x))}$ for $n = 1$.

Theorem E.5 *Assume that $c = 1$ and $f \in C_0^\infty(\overline{B})$. If $n = 1$, then the linearized operator $\Lambda_{(c^2,f)}$ is injective.*

Proof Assume $h, k \in C^\infty(\overline{B})$ are such that $\Lambda_{(c^2,f)}(h,k) = 0$. We will show $h = k = 0$.

Indeed, since $n = 1$ and $c = 1$, the solution u of the equation (5.1) is given by $u(x,t) = \frac{1}{2}(f(x+t) + f(x-t))$. Thus,

$$u_{tt}(x,t) = \frac{1}{2}(f_2(x+t) + f_2(x-t)),$$

where $f_2(x) = f''(x)$.

Choosing $\varphi_\lambda(x) = e^{i\lambda(x-x_0)}$ for an arbitrary $x_0 \in \mathbb{R}^n$, we derive from (5.22) the equation:

$$\int_{\mathbb{R}} h(x) e^{i\lambda(x-x_0)} \int_{\mathbb{R}} \left[f_2(x-t) + f_2(x+t)\right] e^{i\lambda t} dt dx = 0.$$

Taking the integration with respect to λ, we obtain

$$\int_{\mathbb{R}} h(x) \int_{\mathbb{R}} e^{i\lambda(x-x_0)} \left[\int_{\mathbb{R}} f_2(x-t) e^{i\lambda t} dt + \int_{\mathbb{R}} f_2(x+t) e^{i\lambda t} dt\right] dx = 0.$$

Thus,

$$\int_{\mathbb{R}} h(x) \int_{\mathbb{R}} e^{i\lambda(x-x_0)} \left[\int_{\mathbb{R}} f_2(t) e^{i\lambda(x-t)} dt + \int_{\mathbb{R}} f_2(t) e^{i\lambda(t-x)} dt\right] dx = 0.$$

Equivalently,

$$\int_{\mathbb{R}} h(x) \left[\int_{\mathbb{R}} e^{i\lambda(2x-x_0)} d\lambda \int_{\mathbb{R}} f_2(t) e^{-i\lambda t} dt + \int_{\mathbb{R}} e^{i\lambda(-x_0)} d\lambda \int_{\mathbb{R}} f_2(t) e^{i\lambda t} dt\right] dx = 0.$$

The inversion of Fourier transform then gives:

$$\int_{\mathbb{R}} h(x) \left[f_2(2x - x_0) + f_2(x_0)\right] dx = 0.$$

Replacing x_0 by $2x_0$, we obtain

$$\int_{\mathbb{R}} h(x) \left[f_2(2x - 2x_0) + f_2(2x_0)\right] dx = 0.$$

Let $F(x) = f_2(-2x)$, and $c = \int_{\mathbb{R}} h(x)$, we arrive at

$$\int_{\mathbb{R}} h(x) F(x_0 - x) dx = -cF(-x_0).$$

Taking the Fourier transform, we derive the equation:

$$\hat{h}(\lambda)\hat{F}(\lambda) = -c\hat{F}(-\lambda). \tag{5.28}$$

Replacing λ by $-\lambda$, we obtain another equation

$$\hat{h}(-\lambda)\hat{F}(-\lambda) = -c\hat{F}(\lambda). \tag{5.29}$$

Combining (5.29) and (5.28), we then arrive at

$$\hat{h}(-\lambda)\hat{h}(\lambda)\hat{F}(\lambda) = c^2\hat{F}(\lambda).$$

Since F is a nonzero compactly supported function (since f_2 is), $\hat{F}$ is nonzero almost every where. Therefore, $\hat{h}(-\lambda)\hat{h}(\lambda) = c^2$, which implies $(\bar{h} * h)(x) = c_1\delta(x)$. Here c_1 is a constant and $\bar{h}(x) = h(-x)$. Since $h \in C_0^\infty(\mathbb{R})$, one has $\bar{h} * h \in C^\infty(\mathbb{R})$. One thus concludes $c_1 = 0$, and so $h = 0$. Due to Proposition E.1, $k = 0$. This finishes our proof. ∎

CHAPTER VI

SUMMARY

We considered the wave equation model of TAT:

$$\begin{cases} u_{tt}(x,t) - c^2(x) \triangle u(x,t) = 0, \ x \in \mathbb{R}^n, \ t \geq 0, \\ u(x,0) = f(x), \quad u_t(x,0) = 0, \\ u(y,t) = g(y,t), \ \text{for } y \in S, \ t \geq 0, \end{cases} \tag{6.1}$$

and addressed four mathematical different topics of TAT. We summarize here the obtained results:

INVERSION FORMULAS. We translate the equation (6.1) into a nonhomogeneous Helmhotz equation by working in the Fourier domain. Using the Green function for the Helmhotz equation, we obtain a formula for f. However, it involves not only the (Fourier transform of) TAT data g, but also the Neumann data $\frac{\partial u(y,t)}{\partial \nu}$, which is not measured in TAT. We then relate the Neumann data to the TAT data using a range identity. We, hence, derive a family of inversion formulas to reconstruct f from g. This family can be translated into the time domain. It then provides the previously known formulas.

RANGE DESCRIPTION. We translate the problem of range description into the extendibility of the solution $H(x,t)$ of some internal Darboux problem in the unit ball B centered at the origin. The proof of the extendibility boils down to proving that $f(x) = H(x,0)$ vanishes to infinite order on the boundary S of B. We express f in terms of spherical harmonics:

$$f(x) = f(r\theta) = \sum_{m=0}^{\infty} \sum_{k=1}^{d(m)} f_{m,k}(r) r^m Y_{m,k}(\theta),$$

where $r = |x|, |\theta| = 1$ and $\{Y_{m,k}, k = 1, ..., d(m)\}$ is the orthonormal basis in the space of all spherical harmonics of degree m. It is then sufficient to prove that $f_{m,k}(r)$ vanishes to infinite order at $r = 1$. This is proved by combining two facts. Firstly, the spherical mean transform $\mathcal{M}(f)$ of f vanishes to infinite order at $(0, 1)$. Secondly, $\Delta^{(i)} f(y) = 0$ for all $y \in S$ and $i \geq 0$. These two conditions provide an infinite system of linear equations for the derivatives $f^{(j)}_{m,k}(1)$ of $f_{m,k}$. Using a combinatorial argument, we show that this system implies $f^{(j)}_{m,k}(1) = 0$ for all $j \geq 0$.

STABILITY ANALYSIS. We assume that the visibility condition does not hold for a pair of observation surface S and domain of interest Ω. We then show that there is a closed subspace X of $L^2(\Omega)$ such that for all $f \in X$ the TAT data g is smooth. This clearly implies that the reconstruction of the map $\mathcal{T} : f \mapsto g$ is not Lipschitz stable. In order to prove that the reconstruction of $\mathcal{T}$ is not Hölder stable, we make use of the asymptotic behavior of the $s-$numbers (or singular values) of some embeddings between Sobolev spaces.

SPEED DETERMINATION. Work on this problem is still an ongoing project. Here is the list of the partial results we present in this dissertation:

Constant speed. We obtain a formula to recover the constant speed c from the TAT data g. The proof relies on some simple geometric argument that gives a relation between the first arrival time t_0 and last detected time T_0.

Variable speed. A weak local uniqueness result is proved by using the range orthogonality condition. For a given speed $c(x)$ and the TAT data g, we consider a nonzero analytic function $G(\lambda)$ with the zero set $\{\lambda_i\}_{i \geq 0}$. We show that if $\alpha c(x)$ also provides the same TAT data g then $\{\alpha\lambda_i\}_{i\geq 0} \subset \{\lambda_i\}_{i\geq 0}$. This cannot happen if α is close to 1, since the set $\{\lambda_i\}_{i\geq 0}$ does not have any accumulation points.

Characterization of the non-uniqueness. Assume that $c_1(x)$ and $c_2(x) = 1$ provide the same TAT data g. Let $h(x) = c_1^{-2}(x) - c^{-2}(x)_2$, we prove that $h(x) = \Delta V$ for some function $V \in C_0^\infty(\overline{B})$. The proof relies on the fact that there is an analytic family of functions $U_\lambda(x)$ satisfying the elliptic equations:

$$\begin{cases} \nabla\left(\rho_\lambda(x)\nabla U_\lambda(x)\right) + \lambda^2\rho_\lambda(x)h(x)U_\lambda(x) = 0, \\ U|_{\partial\Omega} = 1, \frac{\partial U}{\partial\nu}|_{\partial\Omega} = 0, \end{cases}$$

where $\rho_\lambda(x)$ is an analytic family of positive functions satisfying $\rho_0(x)$ is constant. Expressing the families U_λ, ρ_λ in the series of λ, and identifying the terms of λ^2, we obtain the conclusion.

Linearized problem. Let $\Lambda_{(c^2(x),f)}$ be the linearized operator at $(c^2(x), f(x))$. We prove that if $\Lambda_{(1,f(x))}(h,k) = 0$ then $h(x) = \Delta V(x)$ for some function $V \in C_0^\infty(\overline{B})$. This is similar to the previous result for the nonlinear operator. The proof follows from the orthogonality condition:

$$\int_B h(x)\varphi_\lambda(x)\hat{u}(x,\lambda)dx = 0.$$

Exploiting this relation further, we prove that $\Lambda_{(1,f(x))}$ is injective if $n = 1$.

REFERENCES

[1] Agmon S 1990 A representation theorem for solutions of the Helmholtz equation and resolvent estimates for the Laplacian *Analysis, et cetera* (Boston, MA: Academic Press) pp. 39–76

[2] Agranovsky M, Finch D and Kuchment P 2009 Range condition for a spherical mean transform *Inverse Problems and Imaging* **3** 373–382

[3] Agranovsky M, Kuchment P and Kunyansky L 2009 On reconstruction formulas and algorithms for the thermoacoustic tomography *Photoacoustic Imaging and Spectroscopy* ed L.H. Wang (Boca Raton, FL: CRC Press) chapter 8, pp. 89–101.

[4] Agranovsky M, Kuchment P and Quinto E T 2007 Range descriptions for the spherical mean Radon transform *J. Funct. Anal.* **248** 344–386

[5] Agranovsky M L and Quinto E T 1996 Injectivity sets for the Radon transform over circles and complete systems of radial functions *J. Funct. Anal.* **139** 383–414

[6] Ambartsoumian G and Kuchment P 2006 A range description for the planar circular Radon transform *SIAM J. Math. Anal.* **38** 681–692

[7] Asgeirsson L 1937 Über eine mittelwerteigenschaft von lösungen homogener linearer partieller differentialgleichungen zweiter ordnung mit konstanten koeffizienten *Ann. of Math.* **113**, 321–346

[8] Beylkin G 1983 The fundamental identity for iterated spherical means and the inversion formula for diffraction tomography and inverse scattering *J. Math. Phys.* **24** 1399–1400

[9] Beylkin G 1983 Iterated spherical means in linearized inverse problems *Conference on Inverse Scattering: Theory and Application (Tulsa, Okla.)* (Philadelphia, PA: SIAM) pp. 112–117

[10] Courant R and Hilbert D 1953 *Methods of Mathematical Physics. Vol. I* (New York: Interscience Publishers)

[11] Courant R and Hilbert D 1962 *Methods of Mathematical Physics. Vol. II: Partial Differential Equations* (New York: Interscience Publishers)

[12] Diebold G J, Sun T, and Khan M I 1991 Photoacoustic monopole radiation in one, two, and three dimensions *Phys. Rev. Lett.* **67** (24) 3384–87

[13] Edmunds D E and Triebel H 1996 *Function Spaces, Entropy Numbers, Differential Operators*, (*Cambridge Tracts in Mathematics* vol. 120) (Cambridge: Cambridge Univ. Press)

[14] Epstein C L and Kleiner B 1993 Spherical means in annular regions *Comm. Pure Appl. Math.* **46** 441–451

[15] Evans L C 1998 *Partial Differential Equations* (Providence, RI: American Mathematical Society)

[16] Faridani A, Ritman E L and Smith K T 1992 Examples of local tomography *SIAM J. Appl. Math.* **52** 1193–1198

[17] Faridani A, Ritman E L and Smith K T 1992 Local tomography *SIAM J. Appl. Math.* **52** 459–484

[18] Finch D, Haltmeier M and Rakesh 2007 Inversion of spherical means and the wave equation in even dimensions *SIAM J. Appl. Math.* **68** 392–412

[19] Finch D, Lan I-R and Uhlmann G 2003 Microlocal analysis of the x-ray transform with sources on a curve *Inside Out: Inverse Problems and Applications* (*Math. Sci. Res. Inst. Publ.* vol. 47) (Cambridge: Cambridge Univ. Press) pp. 193–218

[20] Finch D, Patch S K and Rakesh 2004 Determining a function from its mean values over a family of spheres *SIAM J. Math. Anal.* **35** 1213–1240

[21] Finch D and Rakesh 2006 The range of the spherical mean value operator for functions supported in a ball *Inverse Problems* **22** 923–938

[22] Finch D and and Rakesh 2007 The spherical mean value operator with centers on a sphere *Inverse Problems* **23** S37–S49

[23] Finch D and Rakesh 2009. Recovering a function from its spherical mean values in two and three dimensions *Photoacoustic Imaging and Spectroscopy* ed L.H. Wang (Boca Raton, FL: CRC Press) pp. 77-87

[24] Gelfand I M, Gindikin S G and Graev M I 2003 *Selected Topics in Integral Geometry* (*Translations of Mathematical Monographs* vol. 220) (Providence, RI: AMS)

[25] Gohberg I C and Kreĭn M G 1969 *Introduction to the Theory of Linear Nonselfadjoint Operators* (*Translations of Mathematical Monographs* vol. 18) (Providence, RI: AMS)

[26] Greenleaf A and Uhlmann G 1989 Nonlocal inversion formulas for the X-ray transform. *Duke Math. J.* **58** 205–240

[27] Greenleaf A and Uhlmann G 1990 Microlocal techniques in integral geometry *Integral Geometry and Tomography* (*Contemp. Math.* vol. 113) (Providence, RI: AMS) pp. 121–135.

[28] Helgason S 1980 *The Radon Transform* (*Progress in Mathematics* vol. 5) (Boston, Mass.: Birkhäuser)

[29] Hertle A 1988 The identification problem for the constantly attenuated Radon transform. *Math. Z.* **197** 13–19

[30] Hristova Y 2009 Time reversal in thermoacoustic tomography—an error estimate *Inverse Problems* **25** 055008 (14 pp)

[31] Hristova Y, Kuchment P and Nguyen L 2008 Reconstruction and time reversal in thermoacoustic tomography in acoustically homogeneous and inhomogeneous media *Inverse Problems* **24** 055006 (25 pp)

[32] Jin X and Wang L 2009 Theracoustic reconstruction in acoustically heterogrneous media with the aid of ultrasound tomography *Photoacoustic Imaging and Spectroscopy* ed L.H. Wang (Boca Raton, FL: CRC Press) pp. 473-479

[33] Jin Y and Wang L 2006 Thermoacoustic tomography with correction for acoustic speed variations *Physics in Medicine and Biology* **51** 6437–6448.

[34] John F 1981 *Plane Waves and Spherical Means Applied to Partial Differential Equations* (New York: Springer)

[35] Kruger R A, Liu P, Fang Y R and Appledorn C R 1995 Photoacoustic ultrasound (PAUS) reconstruction tomography *Med. Phys.* **22** 1605–09

[36] Kuchment P and Kunyansky L 2008 Mathematics of thermoacoustic tomography *European J. Appl. Math.* **19** 191–224

[37] Kuchment P, Lancaster K and Mogilevskaya L 1995 On local tomography *Inverse Problems* **11** 571–589

[38] Kunyansky L A 2007 Explicit inversion formulae for the spherical mean Radon transform *Inverse Problems* **23** 373–383

[39] Kunyansky L A 2008 Thermoacoustic tomography with detectors on an open curve: an efficient reconstruction algorithm *Inverse Problems* **24** 055021 (18 pp)

[40] Levitan B M 1951 Expansion in Fourier series and integrals with Bessel functions *Uspehi Matem. Nauk (N.S.)* **6** 102–143

[41] Lin V Y and Pinkus A 1994 Approximation of multivariate functions *Advances in Computational Mathematics* (*Ser. Approx. Decompos.* vol. 4) (River Edge, NJ: World Sci. Publ.) pp. 257–265

[42] Louis A K and Quinto E T 2000 Local tomographic methods in Sonar *Surveys on Solution Methods for Inverse Problems* (Vienna: Springer) pp. 147–154

[43] Natterer F 1986 *The Mathematics of Computerized Tomography* (New York: Wiley)

[44] Natterer F and Wübbeling F 2001 *Mathematical Methods in Image Reconstruction* (*Monographs on Mathematical Modeling and Computation* vol 5) (Philadelphia, PA: SIAM)

[45] Nirenberg L 1973 *Lectures on Linear Partial Differential Equations* (Providence, RI: AMS)

[46] Oraevsky A A and Karabutov A A 2002 *Handbook of Optical Biomedical Diagonstics.* ed V. V. Tuchin (Bellingham, WA: SPIE) chapter 10

[47] Oraevsky A A and Karabutov A A 2003 Optoacoustic Tomography *Biomedical Photonics Handbook* ed T. Vo-Dinh (Boca Raton, FL: CRC Press) chapter 34

[48] Palamodov V 2007 Remarks on the general Funk-Radon transform and thermoacoustic tomography *Preprint* http://arxiv.org/abs/math/0701204

[49] Patch S K 2001 Moment conditions indirectly improve image quality *Radon Transforms and Tomography* (*Contemp. Math.* vol. 278) (Providence, RI: AMS) pp. 193–205

[50] Quinto E T 1993 Singularities of the X-ray transform and limited data tomography in $\mathbf{R}^2$ and $\mathbf{R}^3$ *SIAM J. Math. Anal.* **24** 1215–1225

[51] Radon J 1982 Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten (*Proc. Sympos. Appl. Math.* vol. 27) (Providence, RI: AMS) pp. 71–86

[52] Rudin W 1973 *Functional Analysis* (New York: McGraw-Hill)

[53] Solmon D C 1995 The identification problem for the exponential Radon transform *Math. Methods Appl. Sci.* **18** 687–695

[54] Stefanov P and Uhlmann G 2009 Linearizing non-linear inverse problems and an application to inverse backscattering *J. Funct. Anal.* **256** 2842–2866

[55] Stefanov P and Uhlmann G 2009 Thermoacoustic tomography with variable sound speed. *Inverse Problems* **25** 075011 (16 pp)

[56] Steinhauer D 2009 A reconstruction procedure for thermoacoustic tomography in the case of limited boundary data *Preprint* http://arxiv.org/abs/0905.2954

[57] Strichartz R S 2003 *A Guide to Distribution Theory and Fourier Transforms* (River Edge, NJ: World Scientific Publishing)

[58] Tam A C 1986 Applications of photoacoustic sensing techniques *Rev. Mod. Phys.* **58** 381–431

[59] Trimèche K 2001 *Generalized Harmonic Analysis and Wavelet Packets* (Amsterdam: Gordon and Breach Science Publishers)

[60] Vaĭnberg B R 1975 The short-wave asymptotic behavior of the solutions of stationary problems, and the asymptotic behavior as $t \to \infty$ of the solutions of nonstationary problems *Uspehi Mat. Nauk* **30** 3–55

[61] Wang L and Wu H 2007 *Biomedical Optics: Principles and Imaging* (Hoboken, NJ: Wiley)

[62] Wang X, Pang Y, Ku G, Xie X, Stoica G and Wang L 2003 Noninvasive laser-induced photoacoustic tomography for structural and functional *in vivo* imaging of the brain *Nat. Biotechnol.* **21** 803–806

[63] Watson G N 1995 *A Treatise on the Theory of Bessel Functions* Cambridge Mathematical Library (Cambridge: Cambridge Univ. Press)

[64] Xu M and Wang L V 2002 Time-domain reconstruction for thermoacoustic tomography in a spherical geometry *IEEE Trans. Med. Imag.* **21** 814–822

[65] Xu M and Wang L V 2005 Universal back-projection algorithm for photoacoustic computed tomography *Phys. Rev. E* **71** 016706

[66] Xu M and Wang L V 2006 Photoacoustic imaging in biomedicine *Rev. Sci. Instru.* **77** 041101

[67] Xu M, Wang L V, Ambartsoumian G and Kuchment P 2004 Reconstructions in limited view thermoacoustic tomography *Med. Phys.* **31** 724–33

[68] Xu M, Wang L V, Ambartsoumian G and Kuchment P 2009 Limited view thermoacoustic tomography *Photoacoustic Imaging and Spectroscopy* ed L.H. Wang (Boca Raton, FL: CRC Press) pp. 61–73

[69] Xu Y and Wang L 2003 Effects of acoustic heterogeneity in breast thermoacoustic tomography *IEEE Trans. Ultrason. Ferroelectr. Freq. Control* **50** 1134–1146

[70] Zhang J and Anastasio M 2006 Reconstruction of speed-of-sound and electromagnetic absorption distributions in photoacoustic tomography *Proc. SPIE* **6086** (608619) doi:10.1117/12.647665

VITA

Linh Viet Nguyen was born in Quang Binh, a central province of Vietnam. He received a Bachelor of Science in mathematics and computer sciences in 2003 at the National University of Vietnam, Ho Chi Minh City. He received a Master of Science in mathematics at the University of Texas at San Antonio in August 2006. He then started his Ph.D. study in mathematics at Texas A&M University and graduated in August 2010. He will become a postdoctoral fellow in the fall of 2010 at the Mathematical Sciences Research Institute (MSRI) at the University of California, Berkeley. In the spring of 2011, he then assumes an Assistant Professorship at the University of Idaho. His hobbies include photography, reading, and yoga.

He may be reached at linhnguyen1207@yahoo.com, or (from the spring of 2011) Department of Mathematics, University of Idaho, 300 Brink Hall, P.O. Box 441103, Moscow, ID 83844-1103.